The fundamental theorem

Example 22. The four **fundamental subspaces** associated with a matrix A are

$$\mathrm{col}(A), \quad \mathrm{row}(A), \quad \mathrm{null}(A), \quad \mathrm{null}(A^T).$$

Note that $\mathrm{row}(A) = \mathrm{col}(A^T)$. (In particular, we usually write vectors in $\mathrm{row}(A)$ as column vectors.)

Definition 23. $\mathrm{null}(A^T)$ is the **left null space** of A.

Why that name? Recall that, by definition $\boldsymbol{x}$ is in $\mathrm{null}(A) \iff A\boldsymbol{x} = \boldsymbol{0}$.
Likewise, $\boldsymbol{x}$ is in $\mathrm{null}(A^T) \iff A^T\boldsymbol{x} = \boldsymbol{0} \iff \boldsymbol{x}^T A = \boldsymbol{0}$.

[Recall that $(AB)^T = B^T A^T$. In particular, $(A^T\boldsymbol{x})^T = \boldsymbol{x}^T A$, which is what we used in the last equivalence.]

Theorem 24. (Fundamental Theorem of Linear Algebra, Part I)
Let A be an $m \times n$ matrix of **rank** r.

- $\dim \mathrm{col}(A) = r$ (subspace of $\mathbb{R}^m$)
- $\dim \mathrm{row}(A) = r$ (subspace of $\mathbb{R}^n$) $\quad \mathrm{row}(A) = \mathrm{col}(A^T)$
- $\dim \mathrm{null}(A) = n - r$ (subspace of $\mathbb{R}^n$)
- $\dim \mathrm{null}(A^T) = m - r$ (subspace of $\mathbb{R}^m$)

Example 25. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$. Determine bases for all four fundamental subspaces.

Solution. Make sure that, for such a simple matrix, you can see all of these that at a glance!

$\mathrm{col}(A) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$, $\mathrm{row}(A) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$, $\mathrm{null}(A) = \mathrm{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$, $\mathrm{null}(A^T) = \mathrm{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$

Example 26. (important observation) For A as in the previous example, what do you notice about the basis vectors for $\mathrm{row}(A)$ and $\mathrm{null}(A)$? What about $\mathrm{col}(A)$ and $\mathrm{null}(A^T)$?

Solution. The basis vectors for $\mathrm{row}(A)$ and $\mathrm{null}(A)$ are orthogonal! $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$

The same is true for the basis vectors for $\mathrm{col}(A)$ and $\mathrm{null}(A^T)$: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = 0$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = 0$

Vectors in $\mathrm{null}(A)$ are orthogonal to vectors in $\mathrm{row}(A)$.
In short, $\mathrm{null}(A)$ is orthogonal to $\mathrm{row}(A)$.

Why? Suppose that $\boldsymbol{x}$ is in $\mathrm{null}(A)$. That is, $A\boldsymbol{x} = \boldsymbol{0}$.
But think about what $A\boldsymbol{x} = \boldsymbol{0}$ means (row-product rule).
It means that the inner product of every row with $\boldsymbol{x}$ is zero.
But that implies that $\boldsymbol{x}$ is orthogonal to the row space.

Definition 27. As done in the observation above, we say that two subspaces V and W of $\mathbb{R}^n$ are **orthogonal** if and only if every vector in V is orthogonal to every vector in W.
The **orthogonal complement** of W is the space $W^\perp$ of all vectors that are orthogonal to W.

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Exercise. Show that the orthogonal complement is indeed a vector space.

Theorem 28. (Fundamental Theorem of Linear Algebra, Part II)

- $\operatorname{null}(A)$ is orthogonal to $\operatorname{row}(A)$. (both subspaces of $\mathbb{R}^n$)

Note that $\dim \operatorname{null}(A) + \dim \operatorname{row}(A) = n$.
Hence, the two spaces are orthogonal complements.

- $\operatorname{null}(A^T)$ is orthogonal to $\operatorname{col}(A)$.

Again, the two spaces are orthogonal complements.

Note. The second part is just the first part with A replaced by A^T.

Example 29. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$. Check that $\operatorname{null}(A)$ and $\operatorname{row}(A)$ are orthogonal complements.

Solution. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix} \overset{\substack{R_2 - 2R_1 \Rightarrow R_2 \\ R_3 - 3R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -3 \end{bmatrix} \overset{R_3 - \frac{3}{2}R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Hence, $\operatorname{null}(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$, $\operatorname{row}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

The basis vectors are orthogonal because

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 0, \quad \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0.$$

Note. Because $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is orthogonal to both basis vectors, it is orthogonal to every vector in the row space.
Vectors in $\operatorname{row}(A)$ are of the form $\boldsymbol{v} = a \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Then, $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \boldsymbol{v} = a \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$.

Conclusion. Hence, $\operatorname{null}(A)$ and $\operatorname{row}(A)$ are indeed orthogonal spaces.
In fact, $\operatorname{null}(A)$ and $\operatorname{row}(A)$ are orthogonal complements.
That is because $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are orthogonal, hence independent, and thus a basis of all of $\mathbb{R}^3$.

Example 30. (extra) Determine bases for all four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 1 \\ 3 & 6 & 0 & 1 \end{bmatrix}.$$

Verify all parts of the Fundamental Theorem, especially that $\operatorname{null}(A)$ and $\operatorname{row}(A)$ (as well as $\operatorname{null}(A^T)$ and $\operatorname{col}(A)$) are orthogonal complements.

Partial solution. One can almost see that $\operatorname{rank}(A) = 3$. Hence, the dimensions of the fundamental subspaces are ...

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Any serious linear algebra problems are done by a machine. Let us see how to use the open-source computer algebra system **Sage** to do basic computations for us.

Sage is freely available at sagemath.org. Instead of installing it locally (it's huge!) we can conveniently use it in the cloud at cocalc.com from any browser. For short computations, like the one below, you can also just use the input field on our course website.

Sage is built as a **Python** library, so any Python code is valid. Here, we will just use it as a fancy calculator.

Let's revisit Example 29 and let Sage do the work for us:

```
Sage] A = matrix([[1,2,1],[2,4,0],[3,6,0]])
```

```
Sage] A.rref()
```

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Similarly, if we wanted to compute a basis for $\text{null}(A^T)$, we can simply do:

```
Sage] A.transpose().rref()
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Here's some other standard things we might be interested in (compare with Example 13):

```
Sage] A = matrix([[4,0,2],[2,2,2],[1,0,3]])
```

```
Sage] A.eigenvalues()
```

$$[5, 2, 2]$$

```
Sage] A.eigenvectors_right()
```

$$\left[\left(5, \left[\left(1,\ 1,\ \frac{1}{2}\right)\right], 1\right), (2, [(1,\ 0,\ -1), (0,\ 1,\ 0)], 2)\right]$$

```
Sage] A.eigenmatrix_right()
```

$$\left(\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ \frac{1}{2} & -1 & 0 \end{bmatrix}\right)$$

```
Sage] A.rank()
```

3

```
Sage] A.determinant()
```

$$20$$

```
Sage] A.inverse()
```

$$\begin{bmatrix} \frac{3}{10} & 0 & -\frac{1}{5} \\ -\frac{1}{5} & \frac{1}{2} & -\frac{1}{5} \\ -\frac{1}{10} & 0 & \frac{2}{5} \end{bmatrix}$$

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