Example 7. Let us do Gaussian elimination on $A = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$ until we have an echelon form:

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix} \overset{R_2 - 2R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}$$

As last class, the row operation can be encoded by multiplication with an "almost identity matrix" E:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}}_{U}$$

Since $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ (no calculation needed!), this means that

$$A = E^{-1}U = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & -8 \end{bmatrix}.$$

We factored A as the product of a lower and an upper triangular matrix!

$A = LU$ is known as the **LU decomposition** of A.

L is lower triangular, U is upper triangular.

If A is $m \times n$, then L is an invertible lower triangular $m \times m$ matrix, and U is a usual **echelon form** of A. Every matrix A has a LU decomposition (after possibly swapping some rows of A first).

- The matrix U is just the echelon form of A produced during Gaussian elimination.
- The matrix L can be constructed, entry-by-entry, by simply recording the row operations used during Gaussian elimination. (No extra work needed!)

Recall. The **RREF** (row-reduced echelon form) of A is obtained from the echelon form by scaling the pivots to 1, and then eliminating the entries above the pivots. In our example, the RREF of A is the 2×2 identity matrix.

[That's not surprising: A square matrix is invertible if and only if its RREF is the identity matrix. If that isn't obvious to you, think about how you invert a matrix using Gaussian elimination (after augmenting with identity...).]

Example 8. (extra) Determine the LU decomposition of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \overset{R_2 - 3R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ translates into $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$.

Since $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ (no calculation needed!), we therefore have $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$.

Review. Recall the Gauss–Jordan method of computing A^{-1}. Starting with the augmented matrix $[A \mid I]$, we do Gaussian elimination until we obtain the RREF, which will be of the form $[I \mid A^{-1}]$ so that we can read off A^{-1}.

Why does that work? By our discussion, the steps of Gaussian elimination can be expressed by multiplication (on the left) with a matrix B. Only looking at the first part of the augmented matrix, and since the RREF of an invertible matrix is I, we have $BA = I$, which means that we must have $B = A^{-1}$. The other part of the augmented matrix (which is I initially) gets multiplied with $B = A^{-1}$ as well, so that, in the end, it is $BI = A^{-1}$. That's why we can read off A^{-1}!

Armin Straub
straub@southalabama.edu

Review: Eigenvalues and eigenvectors

If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ (and $\boldsymbol{x} \neq \boldsymbol{0}$), then $\boldsymbol{x}$ is an **eigenvector** of A with **eigenvalue** λ (just a number).

Note that for the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to make sense, A needs to be a square matrix (i.e. $n \times n$).

Key observation:

$$\begin{aligned} & A\boldsymbol{x} = \lambda\boldsymbol{x} \\ \Longleftrightarrow\ & A\boldsymbol{x} - \lambda\boldsymbol{x} = \boldsymbol{0} \\ \Longleftrightarrow\ & (A - \lambda I)\boldsymbol{x} = \boldsymbol{0} \end{aligned}$$

This homogeneous system has a nontrivial solution $\boldsymbol{x}$ if and only if $\det(A - \lambda I) = 0$.

To find eigenvectors and eigenvalues of A:

(a) First, find the eigenvalues λ by solving $\det(A - \lambda I) = 0$.

$\det(A - \lambda I)$ is a polynomial in λ, called the **characteristic polynomial** of A.

(b) Then, for each eigenvalue λ, find corresponding eigenvectors by solving $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$.

More precisely, we find a basis of eigenvectors for the λ-**eigenspace** $\operatorname{null}(A - \lambda I)$.

Example 9. $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ has one eigenvector that is "easy" to see. Do you see it?

Solution. Note that $A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Hence, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a 2-eigenvector.

Just for contrast. Note that $A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \neq \lambda\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Hence, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not an eigenvector.

Armin Straub
straub@southalabama.edu