Preparing for the Final

MATH 316 — Linear Algebra II
Monday, April 29

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. The final exam will be comprehensive, that is, it will cover the material of the whole semester.

(a) Do the practice problems for both midterms, as well as the problems below.

(b) Retake both quizzes.

(c) Finally, retake both midterm exams.

Problem 2. Consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$.

(a) Determine the SVD of A.

(b) Determine the best rank 1 approximation of A.

(c) Determine the pseudoinverse of A.

(d) Find the smallest solution to $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(Then, as a mild check, compare its norm to the obvious solution $\boldsymbol{x} = [\begin{array}{ccc} 1 & 1 & 0 \end{array}]^T$.)

Problem 3.

(a) Determine the SVD of $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$.

(b) Determine the best rank 1 approximation of $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$.

(c) Determine the SVD of $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$.

Problem 4. Find the best approximation of $f(x) = x$ on the interval $[0, 4]$ using a function of the form $y = a + b\sqrt{x}$.

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Problem 5. True or false? (As usual, "true" means that the statement is always true.) Explain!

(a) The product of two orthogonal matrices is orthogonal.

(b) $A^T A$ is symmetric for any matrix A.

(c) $A A^T$ is symmetric for any matrix A.

(d) A real $n \times n$ matrix A has real eigenvalues.

(e) The determinant of A is equal to the product of the singular values of A.

(f) The determinant of A is equal to the product of the eigenvalues of A.

(g) If the matrix A is symmetric, then $\langle A\boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{v}, A\boldsymbol{w} \rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

(h) If the matrix A is orthogonal, then $\langle A\boldsymbol{v}, A\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

(i) If $\boldsymbol{v}$ and $\boldsymbol{w}$ are eigenvectors of A with different eigenvalues, then $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \mathbf{0}$.

(j) A is invertible if and only if the only solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$.

(k) An $n \times n$ matrix A has eigenvalue 0 if and only if it has singular value 0.

(l) An $n \times n$ matrix A has eigenvalue 1 if and only if it has singular value 1.

(m) An $n \times n$ matrix A is singular if and only if 0 is an eigenvalue of A.

(n) An $n \times n$ matrix A is singular if and only if 0 is a singular value of A.

(o) Every symmetric real $n \times n$ matrix A is diagonalizable.

(p) Every symmetric real $n \times n$ matrix A is invertible.

(q) A^T has the same eigenvalues as A.

(r) A^T has the same eigenspaces as A.

(s) A^T has the same characteristic polynomial as A.

(t) Every reflection matrix is invertible.

Problem 6.

(a) If A has λ-eigenvalue $\boldsymbol{v}$, then A^3 has [ ].

(b) A is singular if and only if [$\dim \operatorname{null}(A)$].

(c) If $A = \begin{bmatrix} i & 1+2i \\ 3 & 4 \\ 5i & 6-i \end{bmatrix}$, then its conjugate transpose is $A^* =$ [ ].

(d) The norm of the vector $\boldsymbol{v} = \begin{bmatrix} 1-i \\ 2i \end{bmatrix}$ is $\|\boldsymbol{v}\| =$ [ ].

(e) By Euler's identity, $e^{ix} =$ [ ].

(f) What exactly does it mean for a matrix A to have full column rank?

(g) The pseudoinverse of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -7 & 0 \end{bmatrix}$ is $A^+ =$ [ ].

(h) If A is invertible then its pseudoinverse is $A^+ =$ [ ].

(i) If A has full column rank then its pseudoinverse is $A^+ =$ [ ].

(j) Suppose the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has infinitely many solutions $\boldsymbol{x}$.

Which of these solutions is produced by $A^+\boldsymbol{b}$? [ ]

(k) Write down the 2×2 rotation matrix by angle θ.

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