Midterm #2

MATH 316 — Linear Algebra II
Wednesday, Mar 21

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 29 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (8 points) Consider $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

(a) Determine the SVD of A.

(b) Determine the pseudoinverse of A.

Solution.

(a) $A^T A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$ has 10-eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and 0-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

We conclude that $V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$.

$\boldsymbol{u}_1 = \frac{1}{\sigma_1} A \boldsymbol{v}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{20}}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

We cannot obtain $\boldsymbol{u}_2$ in the same way because $\sigma_2 = 0$. Since for every vector $\boldsymbol{u}_2$, $A\boldsymbol{v}_2 = \sigma_2 \boldsymbol{u}_2$, we can choose $\boldsymbol{u}_2$ as we wish, as long as the columns of U are orthonormal in the end.

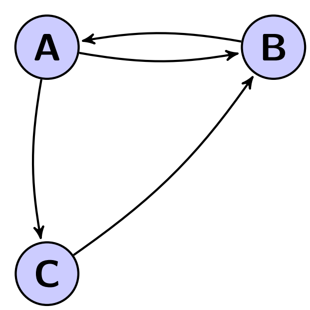
Let's choose $\boldsymbol{u}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ (the only other choice is $\boldsymbol{u}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$). Then, $U = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$.

In summary, $A = U \Sigma V^T$ with $U = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

(b) $A^+ = V \Sigma^+ U^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{10} & \\ & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ □

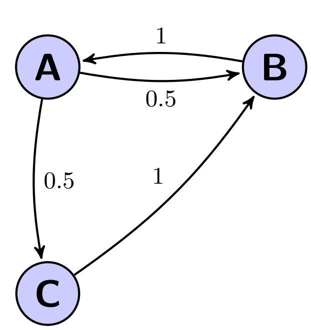
Problem 2. (6 points) Suppose the internet consists of only the three webpages A, B, C which link to each other as indicated in the diagram.

Rank these webpages by computing their PageRank vector.



Solution. Let a_t be the probability that we will be on page A at time t. Likewise, b_t, c_t are the probabilities that we will be on page B or C.

We obtain the following transition behaviour:



$$\begin{bmatrix} a_{t+1} \\ b_{t+1} \\ c_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \cdot a_t + 1 \cdot b_t + 0 \cdot c_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 1 \cdot c_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 0 \cdot c_t \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}\begin{bmatrix} a_t \\ b_t \\ c_t \end{bmatrix}$$

To find the equilibrium state, we again determine an appropriate 1-eigenvector.

The 1-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \\ \frac{1}{2} & 0 & -1 \end{bmatrix}\right)$ which has basis $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

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The corresponding equilibrium state is $\frac{1}{5}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. This is the PageRank vector.

In other words, after browsing randomly for a long time, there is (about) a $\frac{2}{5} = 40\%$ chance to be at page A, a $\frac{2}{5} = 40\%$ chance to be at page B, and a $\frac{1}{5} = 20\%$ chance to be at page C.

We therefore rank A and B highest (tied), and C lowest. □

Problem 3. (3 points) Let A be the 3×3 matrix for reflecting through the plane spanned by the vectors $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Determine an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

Solution. Let $W = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Then A has 1-eigenspace W and -1-eigenspace $W^\perp$. We need orthonormal bases for W and $W^\perp$ in order to write down the diagonalization $A = PDP^T$.

- (basis for W) Since the vectors are already orthogonal, we normalize to find that $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an orthonormal basis for W.
- (basis for $W^\perp$) We can read off that $\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ is an orthonormal basis for $W^\perp$.

In conclusion, we have $A = PDP^T$ with $P = \begin{bmatrix} 2/\sqrt{5} & 0 & -1/\sqrt{5} \\ 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. □

Problem 4. (2 points) Write down a precise definition of what it means for vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m \in \mathbb{R}^n$ to be linearly independent.

Solution. Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m \in \mathbb{R}^n$ are linearly independent if and only if the only solution to

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_m\boldsymbol{v}_m = \mathbf{0}$$

is the trivial one ($x_1 = x_2 = \ldots = x_m = 0$). □

Problem 5. (10 points) Fill in the blanks.

(a) Let A be the 3×3 matrix for an orthogonal projection onto a 2-dimensional subspace.

Then $\det(A) =$ [ ], and the eigenvalues of A are [ ].

(b) If A is $n \times n$, then the product of its singular values equals [ ].

(c) The pseudoinverse of $A = \begin{bmatrix} 4 & 0 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}$ is $A^+ =$ [ ].

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(d) The 2×2 rotation matrix by angle θ is [ ].

(e) If A has full column rank, then its pseudoinverse is given by the formula $A^+ =$ [ ].

(f) Suppose the linear system $A\boldsymbol{x}=\boldsymbol{b}$ has infinitely many solutions $\boldsymbol{x}$.

Which of these solutions is produced by $A^+\boldsymbol{b}$? [ ]

(g) If $A=\begin{bmatrix} 1 & 0\\ 0 & 3 \end{bmatrix}$, then $A^n=$ [ ].

(h) If A is a projection matrix, then $A^{2018}=$ [ ].

(i) If A is a reflection matrix, then $A^{2018}=$ [ ].

(j) If A has SVD $A=U\Sigma V^T$, then A^T has SVD [ ].

Solution.

(a) $\det(A)=0$, and the eigenvalues of A are $0,1,1$.

(b) If A is $n\times n$, then the product of its singular values equals $|\det(A)|$.

(c) The pseudoinverse of $A=\begin{bmatrix} 4 & 0\\ 0 & -2\\ 0 & 0 \end{bmatrix}$ is $A^+=\begin{bmatrix} 1/4 & 0 & 0\\ 0 & -1/2 & 0 \end{bmatrix}$.

(d) The 2×2 rotation matrix by angle θ is $\begin{bmatrix} \cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

(e) If A has full column rank then its pseudoinverse is $A^+=(A^TA)^{-1}A^T$.

(f) The one of smallest norm.

(g) If $A=\begin{bmatrix} 1 & 0\\ 0 & 3 \end{bmatrix}$, then $A^n=\begin{bmatrix} 1 & 0\\ 0 & 3^n \end{bmatrix}$.

(h) If A is a projection matrix, then $A^{2018}=A$. (Because $A^2=A$.)

(i) If A is a reflection matrix, then $A^{2018}=I$. (Because $A^2=I$.)

(j) If A has SVD $A=U\Sigma V^T$, then A^T has SVD $A^T=V\Sigma^TU^T$. $\square$

(extra scratch paper)