Preparing for Midterm #2

MATH 316 — Linear Algebra II
Wednesday, Mar 21

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. Consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$.

(a) Determine the SVD of A.

(b) Determine the pseudoinverse of A.

(c) Find the smallest solution to $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(Then, as a mild check, compare its norm to the obvious solution $\boldsymbol{x} = [\begin{array}{ccc} 1 & 1 & 0 \end{array}]^T$.)

Solution.

(a) $A^T A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ has characteristic polynomial

$$\begin{aligned} \det\left(\begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix} \right) &= 0 - 1 \cdot \det\left(\begin{bmatrix} 2-\lambda & 0 \\ 1 & 1 \end{bmatrix} \right) + (2-\lambda) \det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \right) \\ &= -(2-\lambda) + (2-\lambda) \underbrace{((2-\lambda)(1-\lambda) - 1)}_{=\lambda^2 - 3\lambda + 1} \\ &= (2-\lambda)(\lambda^2 - 3\lambda) = (2-\lambda)\lambda(\lambda - 3). \end{aligned}$$

Hence, the eigenvalues are $0, 2, 3$.

- The 0-eigenspace $\operatorname{null}\left(\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right)$ has basis $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

- The 2-eigenspace $\operatorname{null}\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right)$ has basis $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

- The 3-eigenspace $\operatorname{null}\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \right)$ has basis $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Therefore, $V = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$.

Next, $\boldsymbol{u}_1 = \frac{1}{\sigma_1} A \boldsymbol{v}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{u}_2 = \frac{1}{\sigma_2} A \boldsymbol{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

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Hence, $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

In summary, $A = U\Sigma V^T$ with $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$, $V = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$.

(b) The pseudoinverse of A is

$$A^+ = V\Sigma^+ U^T = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/2 \\ 1/3 & 0 \\ 1/3 & -1/2 \end{bmatrix}.$$

(c) The smallest solution to $A\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is

$$\boldsymbol{x} = A^+ \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/2 \\ 1/3 & 0 \\ 1/3 & -1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 2/3 \\ 1/6 \end{bmatrix}.$$

(For comparison, $\|\boldsymbol{x}\| = \sqrt{11/6} \approx 1.354$ is indeed less than $\|[\begin{matrix} 1 & 1 & 0 \end{matrix}]^T\| = \sqrt{2} \approx 1.414$.) $\square$

Problem 2.

(a) Determine the SVD of $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$.

(b) Determine the SVD of $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$.

Solution.

(a) $A^T A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ has 6-eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and 4-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

We conclude that $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{6} & \\ & 2 \end{bmatrix}$.

$\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{12}} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\boldsymbol{u}_2 = \frac{1}{\sigma_2} A\boldsymbol{v}_2 = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\boldsymbol{u}_3$ needs to be chosen so that the matrix U is orthogonal. To find such a vector, we can start with a random vector like $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and then apply a step of Gram–Schmidt to produce a vector that is orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$:

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$. We normalize this to $\frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$.

Hence, $A = U\Sigma V^T$ with $U = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{6} & \\ & 2 \end{bmatrix}$, $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

(b) $A^T A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}$ has 10-eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and 0-eigenvector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(Hint: We can immediately read off the 0-eigenvector (make sure that's obvious!). It then follows from the spectral theorem that the vector orthogonal to it must be another eigenvector.)

Normalizing, we conclude that $V = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$.

$\boldsymbol{u}_1 = \frac{1}{\sigma_1} A \boldsymbol{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{50}} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

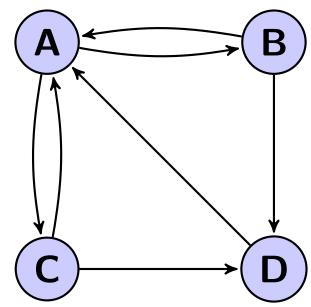
We cannot obtain $\boldsymbol{u}_2$ in the same way because $\sigma_2 = 0$. Since for every vector $\boldsymbol{u}_2$, $A\boldsymbol{v}_2 = \sigma_2 \boldsymbol{u}_2$, we can choose $\boldsymbol{u}_2$ as we wish, as long as the columns of U are orthonormal in the end.

Let's choose $\boldsymbol{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (the only other choice is $\boldsymbol{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$). Then, $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

In summary, $A = U \Sigma V^T$ with $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. □

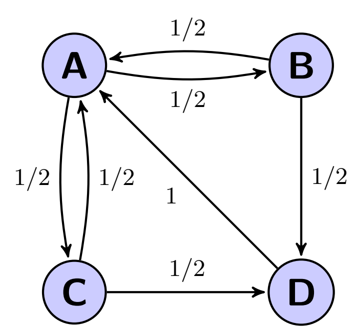
Problem 3. Suppose the internet consists of only the four webpages A, B, C, D which link to each other as indicated in the diagram.

Rank these webpages by computing their PageRank vector.



Solution. Recall that we model a random surfer, who randomly clicks on links. Let a_t be the probability that such a surfer will be on page A at time t. Likewise, b_t, c_t, d_t are the probabilities that the surfer will be on page B, C or D.

The transition probabilities are indicated in the diagram to the right.



$$\begin{bmatrix} a_{t+1} \\ b_{t+1} \\ c_{t+1} \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \cdot a_t + \frac{1}{2} \cdot b_t + \frac{1}{2} \cdot c_t + 1 \cdot d_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 0 \cdot c_t + 0 \cdot d_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 0 \cdot c_t + 0 \cdot d_t \\ 0 \cdot a_t + \frac{1}{2} \cdot b_t + \frac{1}{2} \cdot c_t + 0 \cdot d_t \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}}_{=T} \begin{bmatrix} a_t \\ b_t \\ c_t \\ d_t \end{bmatrix}$$

To find the equilibrium state, we determine an appropriate 1-eigenvector of the transition matrix T.

The 1-eigenspace is $\operatorname{null}(T - 1 \cdot I) = \operatorname{null}\left(\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -1 & 0 & 0 \\ \frac{1}{2} & 0 & -1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \right)$

To compute a basis, we perform Gaussian elimination (details below): $\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -1 & 0 & 0 \\ \frac{1}{2} & 0 & -1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

We conclude that the 1-eigenspace has basis $\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. (Note that its entries add up to $2 + 1 + 1 + 1 = 5$.)

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The corresponding equilibrium state is $\frac{1}{5}\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}$. This is the PageRank vector.

Correspondingly, we rank A the highest, followed by B, C, D which we rank equally.

[In hindsight, can you (at least sort of) see, directly from the diagram, why the PageRank is what it is?]

The full steps of the Gaussian elimination are:

$$\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -1 & 0 & 0 \\ \frac{1}{2} & 0 & -1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \overset{\substack{R_2 + \frac{1}{2}R_1 \Rightarrow R_2 \\ R_3 + \frac{1}{2}R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -\frac{3}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{3}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \overset{\substack{R_3 + \frac{1}{3}R_2 \Rightarrow R_3 \\ R_4 + \frac{2}{3}R_2 \Rightarrow R_4}}{\rightsquigarrow} \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -\frac{3}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \overset{R_4 + R_3 \Rightarrow R_4}{\rightsquigarrow} \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -\frac{3}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{\substack{-1R_1 \Rightarrow R_1 \\ -\frac{4}{3}R_2 \Rightarrow R_2 \\ -\frac{3}{2}R_3 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{\substack{R_1 + \frac{1}{2}R_3 \Rightarrow R_1 \\ R_2 + \frac{1}{3}R_3 \Rightarrow R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_1 + \frac{1}{2}R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This was good practice of elimination! However, notice that we can actually find an eigenvector $\boldsymbol{x}$ with less effort by spelling out the equations: for instance, the second one is just $\frac{1}{2}x_1 - x_2 = 0$. Do that! □

Problem 4. Let A be the 3×3 matrix for reflecting through the plane spanned by the vectors $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Determine an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

Solution. Let $W = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Then A has 1-eigenspace W and -1-eigenspace $W^\perp$. We need orthonormal bases for W and $W^\perp$ in order to write down the diagonalization $A = PDP^T$.

- (basis for W) We apply Gram–Schmidt:

  $$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{5}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$$

  Normalizing, we find that $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{30}}\begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$ is an orthonormal basis for W.

- (basis for $W^\perp$) Since $W = \operatorname{col}\left(\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\right)$, we have $W^\perp = \operatorname{null}\left(\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right)$.

  Solving the system, $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1/2 \end{bmatrix}$, we find that $\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$ is a basis for $W^\perp$. Normalized: $\frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

In conclusion, we have $A = PDP^T$ with $P = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{30} & -1/\sqrt{6} \\ 0 & 5/\sqrt{30} & -1/\sqrt{6} \\ 1/\sqrt{5} & 2/\sqrt{30} & 2/\sqrt{6} \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. □

Problem 5. True or false? (As usual, "true" means that the statement is always true.) Explain!

(a) The product of two orthogonal matrices is orthogonal.

(b) $A^T A$ is symmetric for any matrix A.

(c) $A A^T$ is symmetric for any matrix A.

(d) A real $n \times n$ matrix A has real eigenvalues.

(e) The determinant of A is equal to the product of the singular values of A.

(f) The determinant of A is equal to the product of the eigenvalues of A.

(g) If the matrix A is symmetric, then $\langle A\boldsymbol{v}, \boldsymbol{w}\rangle = \langle \boldsymbol{v}, A\boldsymbol{w}\rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

(h) If the matrix A is orthogonal, then $\langle A\boldsymbol{v}, A\boldsymbol{w}\rangle = \langle \boldsymbol{v}, \boldsymbol{w}\rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

(i) If $\boldsymbol{v}$ and $\boldsymbol{w}$ are eigenvectors of A with different eigenvalues, then $\langle \boldsymbol{v}, \boldsymbol{w}\rangle = \mathbf{0}$.

(j) A is invertible if and only if the only solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$.

(k) An $n \times n$ matrix A has eigenvalue 0 if and only if it has singular value 0.

(l) An $n \times n$ matrix A has eigenvalue 1 if and only if it has singular value 1.

(m) An $n \times n$ matrix A is singular if and only if 0 is an eigenvalue of A.

(n) An $n \times n$ matrix A is singular if and only if 0 is a singular value of A.

(o) Every symmetric real $n \times n$ matrix A is diagonalizable.

(p) Every symmetric real $n \times n$ matrix A is invertible.

(q) A^T has the same eigenvalues as A.

(r) A^T has the same eigenspaces as A.

(s) A^T has the same characteristic polynomial as A.

(t) Every reflection matrix is invertible.

Solution.

(a) True.

If $A^T = A^{-1}$ and $B^T = B^{-1}$, then $(AB)^T = B^T A^T = B^{-1} A^{-1} = (AB)^{-1}$.

(b) True.

(c) True.

(d) False, because this is not true for all matrices. (Take, for instance, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.)

However, by the spectral theorem, a symmetric real $n \times n$ matrix A must have real eigenvalues.

(e) False, but almost true.

Recall that the singular values are all nonnegative, whereas the determinant can be negative.

On the other hand, the absolute value of the determinant of A equals the absolute value of the product of the singular values of A. (Both U and V in $A = U\Sigma V^T$ have determinant ± 1 because they are orthogonal.)

(f) True.

(g) True.

Actually, a matrix A is symmetric if and only if $\langle A\boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{v}, A\boldsymbol{w} \rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

(h) True.

Actually, a matrix A is orthogonal if and only if $\langle A\boldsymbol{v}, A\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

(i) False, because this is not true for all matrices.

However, the statement is true for symmetric matrices by the spectral theorem.

(j) True.

(k) True. Both statements are equivalent to A not being invertible.

(l) False.

(m) True.

(n) True.

(o) True. (That's part of the spectral theorem.)

(p) False.

(q) True. (See Example 102 for this and the next two problems.)

(r) False.

(s) True.

(t) True. (In fact, if A is a reflection matrix, then $A^2 = I$, so that $A^{-1} = A$.) □

Problem 6.

(a) If A has λ-eigenvalue $\boldsymbol{v}$, then A^3 has $\boxed{}$.

(b) A is singular if and only if $\boxed{\dim \operatorname{null}(A) }$.

(c) The eigenvalues of a 5×5 matrix for orthogonally projecting onto a 3-dimensional subspace are $\boxed{}$.

(d) Suppose A is the 3×3 matrix of a reflection through a plane (containing the origin).

Then $\det(A) = \boxed{}$, and the eigenvalues of A are $\boxed{}$.

(e) What exactly does it mean for a matrix A to have full column rank?

(f) Precisely state the spectral theorem.

(g) If A is a reflection matrix, then $A^{-1} = \boxed{}$.

(h) The pseudoinverse of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -7 & 0 \end{bmatrix}$ is $A^+ = \boxed{}$.

(i) If A is invertible then its pseudoinverse is $A^+ = \boxed{}$.

(j) If A has full column rank then its pseudoinverse is $A^+ = \boxed{}$.

(k) Suppose the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has infinitely many solutions $\boldsymbol{x}$.

Which of these solutions is produced by $A^+\boldsymbol{b}$? $\boxed{}$

(l) If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, then what is A^n?

(m) Write down the 2×2 rotation matrix by angle θ.

Solution.

(a) If A has λ-eigenvalue $\boldsymbol{v}$, then A^3 has λ^3-eigenvalue $\boldsymbol{v}$.

(b) A is singular (i.e. not invertible) if and only if $\dim \operatorname{null}(A) > 0$.

(c) The eigenvalues of a 5×5 matrix for orthogonally projecting onto a 3-dimensional subspace are $1, 1, 1, 0, 0$.

(d) Suppose A is the 3×3 matrix of a reflection through a plane (containing the origin).

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Then $\det(A) = -1$, and the eigenvalues of A are $1, 1, -1$.

(e) A matrix A has full column rank if its rank equals the number of columns.

(f) A symmetric (real) matrix A can always be diagonalized. Moreover, all eigenvalues are real and the eigenspaces are orthogonal.

Alternatively: Every symmetric $n \times n$ matrix A can be decomposed as $A = PDP^T$, where D is a diagonal matrix and P is orthogonal.

(g) If A is a reflection matrix, then $A^{-1} = A$ (because $A^2 = I$).

(h) The pseudoinverse of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -7 & 0 \end{bmatrix}$ is $A^+ = \begin{bmatrix} 1/3 & 0 \\ 0 & -1/7 \\ 0 & 0 \end{bmatrix}$.

(i) If A is invertible, then $A^+ = A^{-1}$.

(j) If A has full column rank then its pseudoinverse is $A^+ = (A^T A)^{-1} A^T$.

(k) The one of smallest norm.

(l) $A^n = \begin{bmatrix} 2^n & & \\ & 3^n & \\ & & 4^n \end{bmatrix}$

(m) The 2×2 rotation matrix by angle θ is $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. $\square$