Midterm #1

MATH 316 — Linear Algebra II
Friday, Feb 16

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 28 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (6 points)

(a) Find the least squares solution to $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix}$.

(b) Determine the least squares line for the data points $(0,3), (1,2), (1,0), (2,-1)$.

Solution. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix}$.

(a) Since $A^T A = \begin{bmatrix} 4 & 4 \\ 4 & 6 \end{bmatrix}$ and $A^T \boldsymbol{b} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, so the normal equations $A^T A \hat{\boldsymbol{x}} = A^T \boldsymbol{b}$ are

$$\begin{bmatrix} 4 & 4 \\ 4 & 6 \end{bmatrix} \hat{\boldsymbol{x}} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

Solving, we find that the least squares solution is $\hat{\boldsymbol{x}} = \frac{1}{8} \begin{bmatrix} 6 & -4 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

(b) We need to determine the values a, b for the least squares line $y = a + bx$. The equations $a + bx_i = y_i$ translate into the system

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad \text{that is,} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix}.$$

We have already computed that the least squares solution to that system is $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Hence, the least squares line is $y = 3 - 2x$.

□

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Problem 2. (9 points)

(a) Using Gram–Schmidt, obtain an orthonormal basis for $W = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(b) Determine the orthogonal projection of $\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ onto W.

(c) Determine the QR decomposition of the matrix $\begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$.

(d) Determine a basis for the orthogonal complement $W^{\perp}$.

Solution.

(a) Let $\boldsymbol{w}_1, \boldsymbol{w}_2$ be the vectors spanning W. We then construct an orthonormal basis $\boldsymbol{q}_1, \boldsymbol{q}_2$ using Gram–Schmidt orthonormalization as follows:

- $\boldsymbol{b}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, so that $\boldsymbol{q}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.
- $\boldsymbol{b}_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \cdot \boldsymbol{q}_1 \right) \boldsymbol{q}_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$, so that $\boldsymbol{q}_2 = \frac{1}{\sqrt{8}}\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

We have thus found the orthonormal basis $\frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ for W.

(b) The orthogonal projection of $\boldsymbol{v} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ onto W is

$$(\boldsymbol{v} \cdot \boldsymbol{q}_1)\boldsymbol{q}_1 + (\boldsymbol{v} \cdot \boldsymbol{q}_2)\boldsymbol{q}_2 = \frac{4}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 17 \\ 4 \\ -1 \end{bmatrix}.$$

(Check: the error $\frac{1}{3}(1, -4, 1)^T$ is indeed orthogonal to W.)

(c) From the first part, we know that $Q = \begin{bmatrix} 2/3 & 1/\sqrt{2} \\ 1/3 & 0 \\ 2/3 & -1/\sqrt{2} \end{bmatrix}$.

Hence, $R = Q^T A = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 2\sqrt{2} \end{bmatrix}$.

(d) Clearly, $\dim W^{\perp} = 1$, so that $W^{\perp}$ is spanned by a single vector.

One way to determine vectors in $W^{\perp}$ is to take any vector $\boldsymbol{v}$ (not in W) and project $\boldsymbol{v}$ onto W. The error of that projection then is in $W^{\perp}$.

Without extra computation, we can therefore take the error of the projection in the second part of this problem.

Indeed, the vector $\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 17 \\ 4 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$ is a basis for $W^{\perp}$. □

Problem 3. (3 points) We want to find values for the parameters a, b, c such that $z = a + bx + cy$ best fits some given points $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots$ Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

Solution. The equations $a + bx_i + by_i = z_i$ translate into the system:

$$\underbrace{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ \vdots & \vdots & \vdots \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \end{bmatrix}}_{\boldsymbol{z}}$$

Of course, this is usually inconsistent. To find the best possible a, b, c we compute a least squares solution. □

Problem 4. (2 points) Write down a precise definition of what it means for vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m \in \mathbb{R}^n$ to be linearly independent.

Solution. Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m \in \mathbb{R}^n$ are linearly independent if and only if the only solution to

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_m\boldsymbol{v}_m = \mathbf{0}$$

is the trivial one $(x_1 = x_2 = \ldots = x_m = 0)$. □

Problem 5. (8 points) Fill in the blanks.

(a) If A is a 7×5 matrix with rank 3, then $\dim \operatorname{col}(A) =$ [ ], $\dim \operatorname{row}(A) =$ [ ], $\dim \operatorname{null}(A) =$ [ ].

(b) If B is a 5×3 matrix, then $\operatorname{null}(B)$ is a subspace of [ ] and $\operatorname{col}(B)$ is a subspace of [ ].

(c) $\hat{\boldsymbol{x}}$ is a least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$ if and only if [ ].

(d) $\operatorname{col}(A)$ is the orthogonal complement of [ ]. $\operatorname{null}(A)$ is the orthogonal complement of [ ].

(e) The linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ is orthogonal to [ ].

(f) The projection matrix for orthogonally projecting onto $\operatorname{col}(A)$ is $P =$ [ ].

(g) If P is a projection matrix, then $P^2 =$ [ ].

(h) If W is the space of all solutions to $x_1 + 2x_2 + x_3 - x_4 = 0$, then $\dim W =$ [ ] and $\dim W^\perp =$ [ ].

Solution.

(a) If A is a 7×5 matrix with rank 3, then $\dim \operatorname{col}(A) = 3$, $\dim \operatorname{row}(A) = 3$, $\dim \operatorname{null}(A) = 5 - 3 = 2$.

(b) If B is a 5×3 matrix, then $\operatorname{null}(B)$ is a subspace of $\mathbb{R}^3$ and $\operatorname{col}(B)$ is a subspace of $\mathbb{R}^5$.

(c) $\hat{\boldsymbol{x}}$ is a least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$ if and only if $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$.

(d) $\operatorname{col}(A)$ is the orthogonal complement of $\operatorname{null}(A^T)$. $\operatorname{null}(A)$ is the orthogonal complement of $\operatorname{row}(A)$.

(e) The linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ is orthogonal to $\operatorname{null}(A^T)$.

(f) The projection matrix for orthogonally projecting onto $\operatorname{col}(A)$ is $P = A(A^T A)^{-1} A^T$.

(g) If P is a projection matrix, then $P^2 = P$.

(h) If W is the space of all solutions to $x_1 + 2x_2 + x_3 - x_4 = 0$, then $\dim W = 3$ and $\dim W^\perp = 1$. □

(extra scratch paper)