

Preparing for Midterm #1

Please print your name:

Problem 1.

(a) Using Gram–Schmidt, obtain an orthonormal basis for $W = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(b) Determine the orthogonal projection of $\begin{bmatrix} 2 \\ 6 \\ -1 \\ 3 \end{bmatrix}$ onto W .

(c) Determine the QR decomposition of the matrix $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(d) Determine a basis for the orthogonal complement W^\perp .

Problem 2.

(a) Find the least squares solution to $\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$.

(b) What is the orthogonal projection of $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ onto $W = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right\}$?

(c) Determine the least squares line for the data points $(-2, 1), (-1, 0), (0, 3), (2, 1)$.

(d) Determine the projection matrix P for orthogonally projecting onto W .

Problem 3.

A scientist tries to find the relation between the mysterious quantities x and y .

She measures the following values:

x	1	2	3	4
y	2	5	9	17

(a) Our scientist has reason to expect that y is a linear function of the form $a + bx$. Find the best estimate for the coefficients. [“best” in the least squares sense]

(b) What changes if we suppose that y is a quadratic function of the form $a + bx + cx^2$? Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

Problem 4.

(a) Is it true that $A^T A$ is always symmetric?

(b) When is $A^T A$ a diagonal matrix?

(c) Note that $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Why is it incorrect that the orthogonal projection of $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ onto $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$ is $2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$? Explain!

(d) For which matrices A is it true that $A^{-1} = A^T$?

Problem 5.

(a) We want to find values for the parameters a, b, c such that $y = a + bx + \frac{c}{x}$ best fits some given points $(x_1, y_1), (x_2, y_2), \dots$. Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

(b) We want to find values for the parameters a, b such that $y = (a + bx)e^x$ best fits some given points $(x_1, y_1), (x_2, y_2), \dots$. Set up a linear system such that $[a, b]^T$ is a least squares solution.

(c) We want to find values for the parameters a, b, c such that $z = a + bx - c\sqrt{y}$ best fits some given points $(x_1, y_1, z_1), (x_2, y_2, z_2), \dots$. Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

Problem 6. Let W be the space of all solutions to $x_1 + x_2 + x_3 - x_4 = 0$.

(a) Find a basis for W .

(b) Find a basis for the orthogonal complement W^\perp .

(c) Compute the orthogonal projection of $\mathbf{b} = (1, 1, 1, 1)^T$ onto W^\perp .

(d) Find \mathbf{b}_1 in W and \mathbf{b}_2 in W^\perp such that $\mathbf{b}_1 + \mathbf{b}_2 = (1, 1, 1, 1)^T$.

Problem 7. Suppose that A is a 3×5 matrix of rank 3.

(a) For each of the four fundamental subspaces of A , state which space it is a subspace of.

(b) What are the dimensions of all four fundamental subspaces?

(c) Which fundamental subspaces are orthogonal complements of each other?

(d) For the specific matrix $A = \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 2 & 4 & 0 & 1 & 3 \\ 3 & 6 & 0 & 1 & 4 \end{bmatrix}$, compute a basis for each fundamental subspace.

(e) Observe that $\text{rank}(A) = 3$. Then, verify that all your predictions made in the first three parts do in fact hold.