Preparing for Midterm #1

MATH 316 — Linear Algebra II
Friday, Feb 16

Please print your name:

Problem 1.

(a) Using Gram–Schmidt, obtain an orthonormal basis for $W = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(b) Determine the orthogonal projection of $\begin{bmatrix} 2 \\ 6 \\ -1 \\ 3 \end{bmatrix}$ onto W.

(c) Determine the QR decomposition of the matrix $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(d) Determine a basis for the orthogonal complement $W^\perp$.

Solution.

(a) Let $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ be the vectors spanning W. We then construct an orthonormal basis $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ using Gram–Schmidt orthonormalization as follows:

- $\boldsymbol{b}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, so that $\boldsymbol{q}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

- $\boldsymbol{b}_2 = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} \cdot \boldsymbol{q}_1 \right) \boldsymbol{q}_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, so that $\boldsymbol{q}_2 = \frac{1}{3} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$.

- $\boldsymbol{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \cdot \boldsymbol{q}_1 \right) \boldsymbol{q}_1 - \left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \cdot \boldsymbol{q}_2 \right) \boldsymbol{q}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{9} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 4 \end{bmatrix}$, so that $\boldsymbol{q}_3 = \frac{1}{\sqrt{18}} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 4 \end{bmatrix}$.

We have thus found the orthonormal basis $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{18}} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 4 \end{bmatrix}$ for W.

(b) The orthogonal projection of $\boldsymbol{v} = \begin{bmatrix} 2 \\ 6 \\ -1 \\ 3 \end{bmatrix}$ onto W is

$$(\boldsymbol{v} \cdot \boldsymbol{q}_1)\boldsymbol{q}_1 + (\boldsymbol{v} \cdot \boldsymbol{q}_2)\boldsymbol{q}_2 + (\boldsymbol{v} \cdot \boldsymbol{q}_3)\boldsymbol{q}_3 = 6 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{5}{9} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \frac{11}{18} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 6 \\ 1/2 \\ 3 \end{bmatrix}.$$

(c) From the first part, we know that $Q = \begin{bmatrix} 0 & 2/3 & -1/\sqrt{18} \\ 1 & 0 & 0 \\ 0 & 2/3 & -1/\sqrt{18} \\ 0 & 1/3 & 4/\sqrt{18} \end{bmatrix}$.

Hence, $R = Q^T A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2/3 & 0 & 2/3 & 1/3 \\ -1/\sqrt{18} & 0 & -1/\sqrt{18} & 4/\sqrt{18} \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 3 & 5/3 \\ 0 & 0 & 2/\sqrt{18} \end{bmatrix}$.

Armin Straub
straub@southalabama.edu

(d) Clearly, $\dim W^{\perp} = 1$, so that $W^{\perp}$ is spanned by a single vector.

One way to determine vectors $W^{\perp}$ is to take any vector $\boldsymbol{v}$ (not in W) and project $\boldsymbol{v}$ onto W. The error of that projection then is in $W^{\perp}$.

Without extra computation, we can therefore take the error of the projection in the second part of this problem.

Indeed, the vector $\begin{bmatrix} 2 \\ 6 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 6 \\ 1/2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \\ -3/2 \\ 0 \end{bmatrix}$ is a basis for $W^{\perp}$.

□

Problem 2.

(a) Find the least squares solution to $\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$.

(b) What is the orthogonal projection of $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ onto $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right\}$?

(c) Determine the least squares line for the data points $(-2,1), (-1,0), (0,3), (2,1)$.

(d) Determine the projection matrix P for orthogonally projecting onto W.

Solution. Let $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 3 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$.

(a) We compute $A^T A = \begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix}$ and $A^T \boldsymbol{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, so the normal equations $A^T A \hat{\boldsymbol{x}} = A^T \boldsymbol{b}$ are

$$\begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix} \hat{\boldsymbol{x}} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

Solving, we find that the least squares solution is $\hat{\boldsymbol{x}} = \frac{1}{35} \begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 9 \\ 1 \end{bmatrix}$.

(b) The orthogonal projection of $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ onto W is $A\hat{\boldsymbol{x}} = \frac{1}{7} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 9 \\ 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 7 \\ 8 \\ 9 \\ 11 \end{bmatrix}$.

Check. The error $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix} - \frac{1}{7} \begin{bmatrix} 7 \\ 8 \\ 9 \\ 11 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 0 \\ -8 \\ 12 \\ -4 \end{bmatrix}$ is orthogonal to both $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -1 \\ 0 \\ 2 \end{bmatrix}$.

(c) We need to determine the values a, b for the least squares line $y = a + bx$. The equations $a + bx_i = y_i$ translate into the system

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad \text{that is,} \quad \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}.$$

We have already computed that the least squares solution to that system is $\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 9 \\ 1 \end{bmatrix}$.

Hence, the least squares line is $y = \frac{9}{7} + \frac{1}{7}x$.

(d) The projection matrix is $P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \frac{1}{35} \begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 21 & 14 & 7 & -7 \\ 14 & 11 & 8 & 2 \\ 7 & 8 & 9 & 11 \\ -7 & 2 & 11 & 29 \end{bmatrix}$.

□

Problem 3. A scientist tries to find the relation between the mysterious quantities x and y.

She measures the following values:

x	1	2	3	4
y	2	5	9	17

(a) Our scientist has reason to expect that y is a linear function of the form $a + bx$. Find the best estimate for the coefficients. ["best" in the least squares sense]

(b) What changes if we suppose that y is a quadratic function of the form $a + bx + cx^2$? Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

Solution.

(a) If we had $y = a + bx$ exactly, then we could find a, b by solving the system

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 5 \\ 9 \\ 17 \end{bmatrix}}_{\boldsymbol{y}}.$$

To find the least squares estimate, we solve the normal equations $A^TA\begin{bmatrix} a \\ b \end{bmatrix} = A^T\boldsymbol{y}$.

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix} \text{ and } A^T\boldsymbol{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 9 \\ 17 \end{bmatrix} = \begin{bmatrix} 33 \\ 107 \end{bmatrix}.$$

We solve $\begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 33 \\ 107 \end{bmatrix}$ to find $\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix} \begin{bmatrix} 33 \\ 107 \end{bmatrix} = \begin{bmatrix} -4 \\ 49/10 \end{bmatrix}$.

Hence, $a = -4$ and $b = 4.9$.

(b) Again, if we had $y = a + bx + cx^2$ exactly, then we could find a, b, c by solving the system

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 5 \\ 9 \\ 17 \end{bmatrix}}_{\boldsymbol{y}}.$$

We find the best fit by instead computing a least squares solution.

Extra. Now, it becomes a bit painful by hand (ask Sage for help!). The normal equations $A^TA\begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^T\boldsymbol{y}$ are:

$$\begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 33 \\ 107 \\ 375 \end{bmatrix}.$$

Solving this system, we find $a = 2.25$, $b = -1.35$ and $c = 1.25$. □

Armin Straub
straub@southalabama.edu

Problem 4.

(a) Is it true that $A^T A$ is always symmetric?

(b) When is $A^T A$ a diagonal matrix?

(c) Note that $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Why is it incorrect that the orthogonal projection of $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ onto span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is $2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$? Explain!

(d) For which matrices A is it true that $A^{-1} = A^T$?

Solution.

(a) Yes, $A^T A$ is always symmetric: $(A^T A)^T = A^T (A^T)^T = A^T A$

(b) $A^T A$ is a diagonal matrix if and only if the columns of A are orthogonal.

(c) The vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ are not an orthogonal basis for the span.

(d) For a square matrix, $A^{-1} = A^T$ if and only if $A^T A = I$. Hence, $A^{-1} = A^T$ if and only if A is a square matrix with orthonormal columns (that's what we call an orthogonal matrix). □

Problem 5.

(a) We want to find values for the parameters a, b, c such that $y = a + bx + \frac{c}{x}$ best fits some given points (x_1, y_1), $(x_2, y_2), \ldots$ Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

(b) We want to find values for the parameters a, b such that $y = (a + bx)e^x$ best fits some given points (x_1, y_1), $(x_2, y_2), \ldots$ Set up a linear system such that $[a, b]^T$ is a least squares solution.

(c) We want to find values for the parameters a, b, c such that $z = a + bx - c\sqrt{y}$ best fits some given points (x_1, y_1, z_1), $(x_2, y_2, z_2), \ldots$ Set up a linear system such that $[a, b, c]^T$ is a least squares solution.

Solution.

(a) The equations $a + bx_i + c/x_i = y_i$ translate into the system:

$$\underbrace{\begin{bmatrix} 1 & x_1 & 1/x_1 \\ 1 & x_2 & 1/x_2 \\ 1 & x_3 & 1/x_3 \\ \vdots & \vdots & \vdots \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix}}_{\boldsymbol{y}}$$

Of course, this is usually inconsistent. To find the best possible a, b, c we compute a least squares solution.

(b) The equations $(a + bx_i)e^{x_i} = y_i$ translate into the system:

$$\underbrace{\begin{bmatrix} e^{x_1} & x_1 e^{x_1} \\ e^{x_2} & x_2 e^{x_2} \\ e^{x_3} & x_3 e^{x_3} \\ \vdots & \vdots \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix}}_{\boldsymbol{y}}$$

Armin Straub
straub@southalabama.edu

Of course, this is usually inconsistent. To find the best possible a, b we compute a least squares solution.

(c) The equations $a + bx_i - c\sqrt{y_i} = z_i$ translate into the system:

$$\underbrace{\begin{bmatrix} 1 & x_1 & -\sqrt{y_1} \\ 1 & x_2 & -\sqrt{y_2} \\ 1 & x_3 & -\sqrt{y_3} \\ \vdots & \vdots & \vdots \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \end{bmatrix}}_{\boldsymbol{z}}$$

Of course, this is usually inconsistent. To find the best possible a, b, c we compute a least squares solution. □

Problem 6. Let W be the space of all solutions to $x_1 + x_2 + x_3 - x_4 = 0$.

(a) Find a basis for W.

(b) Find a basis for the orthogonal complement $W^\perp$.

(c) Compute the orthogonal projection of $\boldsymbol{b} = (1, 1, 1, 1)^T$ onto $W^\perp$.

(d) Find $\boldsymbol{b}_1$ in W and $\boldsymbol{b}_2$ in $W^\perp$ such that $\boldsymbol{b}_1 + \boldsymbol{b}_2 = (1, 1, 1, 1)^T$.

Solution. Note that $W = \text{null}(A)$ for the matrix $A = [\,1 \;\; 1 \;\; 1 \;\; -1\,]$.

(a) A is already in RREF, so we can read off that $W = \text{null}(A)$ consists of the vectors $\begin{bmatrix} -s_1 - s_2 + s_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}$.

Hence, a basis for W is: $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(b) Recall that the orthogonal complement of $\text{null}(A)$ is $\text{row}(A)$.

Hence, a basis for $W^\perp$ is: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$. (Note how this vector is indeed orthogonal to all basis vectors of W.)

(c) Since $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ is an orthogonal basis for $W^\perp$, the projection is $\frac{\boldsymbol{b} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}} \boldsymbol{v} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$.

(d) Note that this meas that $\boldsymbol{b}_1$ is the orthogonal projection of $\boldsymbol{b} = (1, 1, 1, 1)^T$ onto W, and $\boldsymbol{b}_2$ is the the orthogonal projection of $\boldsymbol{b}$ onto $W^\perp$.

The easiest way to compute these is to note that, from the previous part, we already know $\boldsymbol{b}_2 = \frac{\boldsymbol{b} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}} \boldsymbol{v} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$.

Consequently, $\boldsymbol{b}_1 = \boldsymbol{b} - \boldsymbol{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$.

[To check, we can verify that $\boldsymbol{b}_1$ is indeed in W by plugging it into the defining equation.] □

Problem 7. Suppose that A is a 3×5 matrix of rank 3.

(a) For each of the four fundamental subspaces of A, state which space it is a subspace of.

(b) What are the dimensions of all four fundamental subspaces?

(c) Which fundamental subspaces are orthogonal complements of each other?

Armin Straub
straub@southalabama.edu

(d) For the specific matrix $A = \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 2 & 4 & 0 & 1 & 3 \\ 3 & 6 & 0 & 1 & 4 \end{bmatrix}$, compute a basis for each fundamental subspace.

(e) Observe that $\operatorname{rank}(A) = 3$. Then, verify that all your predictions made in the first three parts do in fact hold.

Solution.

(a) $\operatorname{col}(A)$ and $\operatorname{null}(A^T)$ are subspaces of $\mathbb{R}^3$, while $\operatorname{row}(A)$ and $\operatorname{null}(A)$ are subspaces of $\mathbb{R}^5$.

(b) $\dim \operatorname{col}(A) = 3$, $\dim \operatorname{row}(A) = 3$, $\dim \operatorname{null}(A) = 5 - 3 = 2$, $\dim \operatorname{null}(A^T) = 3 - 3 = 0$.

(c) $\operatorname{col}(A)$ and $\operatorname{null}(A^T)$ are orthogonal complements of each other.

Also, $\operatorname{row}(A)$ and $\operatorname{null}(A)$ are orthogonal complements of each other.

(d) Gaussian elimination:

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 2 & 4 & 0 & 1 & 3 \\ 3 & 6 & 0 & 1 & 4 \end{bmatrix} \overset{\substack{R_2 - 2R_1 \Rightarrow R_2 \\ R_3 - 3R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 0 & -2 & -5 & -5 \\ 0 & 0 & -3 & -8 & -8 \end{bmatrix} \overset{R_3 - \frac{3}{2}R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 0 & -2 & -5 & -5 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\overset{\substack{-\frac{1}{2}R_2 \Rightarrow R_2 \\ -2R_3 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \overset{\substack{R_1 - 3R_3 \Rightarrow R_1 \\ R_2 - \frac{5}{2}R_3 \Rightarrow R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \overset{R_1 - R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Hence, we can read off the bases:

$\operatorname{col}(A)$ has basis $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

(Knowing that $\dim \operatorname{col}(A) = 3$, so that $\operatorname{col}(A) = \mathbb{R}^3$, we could have also just written down the standard basis.)

$\operatorname{row}(A)$ has basis $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 0 \\ 1 \\ 4 \end{bmatrix}$.

$\operatorname{null}(A)$ consists of the vectors $\begin{bmatrix} -2s_1 - s_2 \\ s_1 \\ 0 \\ -s_2 \\ s_2 \end{bmatrix}$ and so has basis $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

$\operatorname{null}(A^T)$ has dimension 0 (contains only the zero vector), and so has an empty basis (consisting of 0 vectors).

(e) The rank is the number of pivots, which is indeed 3 (also equals $\dim \operatorname{col}(A)$ and $\dim \operatorname{row}(A)$).

We predicted all the dimensions accurately.

□