1 Preparing for Midterm 1

- These problems are taken from the lectures to help you prepare for our upcoming midterm exam. You can find solutions to all of these in the lecture sketches.
- Additional, more exam-like, practice problems are also posted to our website.

Example 1. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \boxed{}$

Example 2. Determine a basis for the orthogonal complement of $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

Example 3. Determine a basis for the orthogonal complement of $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}$.

Theorem 4. **(Fundamental Theorem of Linear Algebra, Part I)**
Let A be an $m \times n$ matrix of **rank** r.

- $\dim \operatorname{col}(A) = \boxed{}$ (subspace of $\boxed{}$)
- $\dim \operatorname{row}(A) = \boxed{}$ (subspace of $\boxed{}$)
- $\dim \operatorname{null}(A) = \boxed{}$ (subspace of $\boxed{}$)
- $\dim \operatorname{null}(A^T) = \boxed{}$ (subspace of $\boxed{}$)

Example 5. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$. Determine bases for all four fundamental subspaces.

Theorem 6. **(Fundamental Theorem of Linear Algebra, Part II)**

- $\operatorname{null}(A)$ is the orthogonal complement of $\boxed{}$.
- $\operatorname{null}(A^T)$ is the orthogonal complement of $\boxed{}$.

Example 7. Determine bases for all four fundamental subspaces of

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix} \quad \text{as well as} \quad \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 1 \\ 3 & 6 & 0 & 1 \end{bmatrix}.$$

Verify all parts of the Fundamental Theorem.

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Theorem 8. $A\boldsymbol{x} = \boldsymbol{b}$ is consistent $\iff$ $\boldsymbol{b}$ is orthogonal to $\boxed{}$

Example 9. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$. For which $\boldsymbol{b}$ does $A\boldsymbol{x} = \boldsymbol{b}$ have a solution?

Theorem 10. $\hat{\boldsymbol{x}}$ is a least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$ $\iff$ $\boxed{}$

Example 11. Find the least squares solution to $\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Example 12. Find the least squares solution to $\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Example 13. Find the least squares solution to $\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$.

The (orthogonal) projection $\hat{\boldsymbol{b}}$ of $\boldsymbol{b}$ onto $\operatorname{col}(A)$ is $\hat{\boldsymbol{b}} = \boxed{}$.

Example 14. What is the orthogonal projection of $\begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$ onto $\operatorname{span}\left\{\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right\}$?

Example 15.

(a) What is the orthogonal projection of $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ onto $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}$?

(b) What is the orthogonal projection of $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ onto $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$?

Example 16. Determine the line that best fits the data points $(2,1), (5,2), (7,3), (8,3)$.

Example 17. A car rental company wants to predict the annual maintenance cost y (in 100USD/year) of a car using the age x (in years) of that car (as an explanatory variable). Based on the observations $(x,y) = (2,1), (5,2), (7,3), (8,3)$, predict the cost for a 4.5 year old car (using linear regression).

Example 18. Set up a linear system to find values for the parameters a, b, c such that $z = a + bx + cy$ best fits some given points $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots$

Example 19. Set up a linear system to find values for the parameters a, b, c that result in the quadratic curve $y = a + bx + cx^2$ that best fits some given points $(x_1, y_1), (x_2, y_2), \ldots$

Example 20. Find values for the parameters a, b, c that result in the quadratic curve $y = a + bx + cx^2$ that best fits the points $(0,1), (1,2), (2,3), (3,-4), (4,-7), (5,-12)$. If working by hand, just set up the system.

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Example 21. What is the orthogonal projection of $\boldsymbol{v} = (1,2,3)^T$ onto $\boldsymbol{w} = (1,1,1)^T$?

The (orthogonal) projection of $\boldsymbol{v}$ onto $\boldsymbol{w}$ is $\boxed{}$.

$P = \boxed{}$ is the **projection matrix** for projecting onto $\mathrm{col}(A)$.

Example 22.

(a) Determine the projection matrix P for projecting onto $W = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(b) Determine the projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto W using the projection matrix.

Example 23. If P is a projection matrix, then what is P^2?

Example 24. True or false? If P is the matrix for projecting onto W, then $W = \mathrm{col}(P)$.

Example 25.

(a) What is the matrix P for projecting onto $W = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$?

(b) Using the projection matrix, project $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ onto $W = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Example 26. Give precise definitions for the following:

- Vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent.
- Vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are a basis for V.

Example 27. Are the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ an orthogonal basis for $\mathbb{R}^3$? Is it orthonormal?

If not, normalize the vectors to produce an orthonormal basis.

If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthogonal basis of V, and $\boldsymbol{w}$ is in V, then $$\boldsymbol{w} = c_1 \boldsymbol{v}_1 + \ldots + c_n \boldsymbol{v}_n \qquad \text{with} \quad c_j = \boxed{}.$$

Example 28. Express $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ in terms of the basis $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Example 29. Find an orthogonal basis for $W = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

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Example 30. Using Gram–Schmidt, find an orthogonal basis for $W = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Example 31. Find an orthonormal basis for $W = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(Gram–Schmidt orthonormalization)

Given a basis $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots$ for W, produce an orthonormal basis $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots$ for W.

- $\boldsymbol{q}_1 =$ [ ]
- $\boldsymbol{q}_2 =$ [ ]
- $\boldsymbol{q}_3 =$ [ ]
- $\boldsymbol{q}_4 = \ldots$

Example 32. A matrix A satisfies $A^T A = I$ if and only if ...

Example 33. Determine the QR decomposition of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Example 34. Find the QR decomposition of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

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