Review. Fourier series

Review. $[0,1]$ is uncountable, but the rational numbers in $[0,1]$ are countable

Linear approximations to arbitrary functions

- **(Calculus 1)** Let $f(x)$ be any (nice; i.e. differentiable) function $\mathbb{R}\to\mathbb{R}$.
  Suppose the best linear approximation to $f(x)$ at $x=a$ is $f(x)\approx f(a)+m(x-a)$.
  Geometrically, $x\mapsto f(a)+m(x-a)$ describes the tangent line to $f(x)$ at $x=a$.
  In Calculus 1, you learn that the crucial quantity m gives rise to the **derivative**, $m=f'(a)$, and you learn how to compute it, given a function $f(x)$.
- **(Calculus 3)** Let $\boldsymbol{f}(\boldsymbol{x})$ be any (nice) function $\mathbb{R}^n\to\mathbb{R}^m$.
  Now, the best linear approximation to $\boldsymbol{f}(\boldsymbol{x})$ at $\boldsymbol{x}=\boldsymbol{a}$ is $\boldsymbol{f}(\boldsymbol{x})\approx\boldsymbol{f}(\boldsymbol{a})+M(\boldsymbol{x}-\boldsymbol{a})$.
  Here, M is a $m\times n$ matrix. (Why these dimensions?!)
  In Calculus 3, you again learn how to compute the quantity $M=D\boldsymbol{f}(\boldsymbol{a})$ (the **derivative** or **Jacobian matrix**), given a function $f(x)$. Indeed:

$$\boldsymbol{f}(\boldsymbol{x})=\begin{bmatrix} f_1(x_1,\ldots,x_n)\\ \vdots\\ f_m(x_1,\ldots,x_n)\end{bmatrix},\quad D\boldsymbol{f}=\begin{bmatrix} \frac{\partial}{\partial x_1}f_1 & \cdots & \frac{\partial}{\partial x_n}f_1\\ \vdots & & \vdots\\ \frac{\partial}{\partial x_1}f_m & \cdots & \frac{\partial}{\partial x_n}f_m\end{bmatrix}$$

Comment. One important point of this discussion is that, through our linear algebra glasses, the transition from Calculus 1 to Calculus 3 is exactly as expected.

Example 184. Determine the best linear approximation to $\boldsymbol{f}(x,y)=\begin{bmatrix} x^2+y^2\\ xy+1\end{bmatrix}$ at $\boldsymbol{x}=\begin{bmatrix} 2\\ 1\end{bmatrix}$.

Solution. The Jacobian matrix at $\boldsymbol{x}=\begin{bmatrix} x\\ y\end{bmatrix}$ is

$$D\boldsymbol{f}(x,y)=\begin{bmatrix} \frac{\partial}{\partial x}(x^2+y^2) & \frac{\partial}{\partial y}(x^2+y^2)\\ \frac{\partial}{\partial x}(xy+1) & \frac{\partial}{\partial x}(xy+1)\end{bmatrix}=\begin{bmatrix} 2x & 2y\\ y & x\end{bmatrix},\quad D\boldsymbol{f}(2,1)=\begin{bmatrix} 4 & 2\\ 1 & 2\end{bmatrix}.$$

Hence, the best linear approximation to $\boldsymbol{f}(x,y)$ at $\boldsymbol{x}=\begin{bmatrix} 2\\ 1\end{bmatrix}$ is

$$\boldsymbol{f}(x,y)\approx\boldsymbol{f}(2,1)+\begin{bmatrix} 4 & 2\\ 1 & 2\end{bmatrix}\left(\begin{bmatrix} x\\ y\end{bmatrix}-\begin{bmatrix} 2\\ 1\end{bmatrix}\right)=\begin{bmatrix} 5\\ 3\end{bmatrix}+\begin{bmatrix} 4 & 2\\ 1 & 2\end{bmatrix}\begin{bmatrix} x-2\\ y-1\end{bmatrix}.$$

Comment. For most purposes, this is the nicest form to write the linear approximation. If we want to multiply out the matrix-vector product, we get

$$\boldsymbol{f}(x,y)\approx\begin{bmatrix} 4x+2y-5\\ x+2y-1\end{bmatrix}.$$

Note how we solved two independent problems at once: approximating x^2+y^2 at $(x,y)=(2,1)$, and approximating $xy+1$ at $(x,y)=(2,1)$.

Armin Straub
straub@southalabama.edu

Example 185. (extra) For $\boldsymbol{f}(r,\ \theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$, determine the Jacobian matrix (i.e. the derivative) and its determinant.

Solution. The Jacobian matrix is $\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$.

Its determinant is $\det\left(\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}\right) = r\cos^2\theta + r\sin^2\theta = r$.

Example 186. If you have taken Calculus 3, you have learned about **substitution** in multiple integrals. To make the change of variables $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{u})$, that is, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} g_1(u_1, u_2) \\ g_2(u_1, u_2) \end{bmatrix}$, we have

$$\int\int_R f(\boldsymbol{x})\mathrm{d}x_2\mathrm{d}x_1 = \int\int_G f(\boldsymbol{g}(\boldsymbol{u}))|D\boldsymbol{g}(\boldsymbol{u})|\mathrm{d}u_2\mathrm{d}u_1$$

- where G is the region in the u_1u_2-plane corresponding to the region R in the x_1x_2-plane,
- and $|D\boldsymbol{g}(\boldsymbol{u})| = \det\begin{bmatrix} \frac{\partial}{\partial u_1}g_1 & \frac{\partial}{\partial u_2}g_1 \\ \frac{\partial}{\partial u_1}g_2 & \frac{\partial}{\partial u_2}g_2 \end{bmatrix}$ is the **Jacobian determinant**.

Comment. Have another look at Example 92. In that example, we observed that $\det(A)$ measures by much a little volume (or area) is scaled under $\boldsymbol{x} \mapsto A\boldsymbol{x}$. Hence, it makes perfect sense that the formula for substitution needs to take into account how much the function $\boldsymbol{g}$ changes volumes. Locally, $\boldsymbol{u} \mapsto \boldsymbol{g}(\boldsymbol{u})$ is approximated (up to a shift) by multiplication with $D\boldsymbol{g}$, and so the change in volume is measured by the Jacobian determinant.

For instance. From your computation in the previous example, it follows that, using **polar coordinates** $x = r\cos\theta$, $y = r\sin\theta$, we have the familiar formula

$$\int\int_R f(x, y)\mathrm{d}y\mathrm{d}x = \int\int_G f(r\cos\theta, r\sin\theta)r\mathrm{d}r\mathrm{d}\theta.$$

Armin Straub
straub@southalabama.edu