Review 166. fundamental theorem of algebra

Example 167. What is the norm of the vector $\begin{bmatrix} 1-i \\ 2+3i \end{bmatrix}$?

Solution. $\left\| \begin{bmatrix} 1-i \\ 2+3i \end{bmatrix} \right\|^2 = \begin{bmatrix} 1+i & 2-3i \end{bmatrix} \begin{bmatrix} 1-i \\ 2+3i \end{bmatrix} = |1-i|^2 + |2+3i|^2 = 2+13$. Hence, $\left\| \begin{bmatrix} 1-i \\ 2+3i \end{bmatrix} \right\| = \sqrt{15}$.

Example 168. Determine A^* if $A = \begin{bmatrix} 2 & 1-i \\ 3+2i & i \end{bmatrix}$.

Solution. $A^* = \begin{bmatrix} 2 & 3-2i \\ 1+i & -i \end{bmatrix}$

Example 169. Find a unitary matrix Q whose first column is a multiple of $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

Solution. (sketch) We need to find a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ such that $\begin{bmatrix} 1 \\ i \end{bmatrix}^* \begin{bmatrix} a \\ b \end{bmatrix} = a - ib = 0$. Choose, say, $a = i$, $b = 1$.

This leads to the unitary matrix $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$. Indeed, $Q^* Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Function spaces

Recall the following:

- We call objects **vectors** if they can be added and scaled (subject to the usual laws).
- A set of vectors is a **vector space** if it is closed under addition and scaling.

  In other words, vector spaces are spans.

We will now discuss spaces of vectors, where the vectors are functions.

Why? Just one example why it is super useful to apply our linear algebra machinery to functions: we discussed the **distance** between vectors and how to find vectors closest to interesting subspaces (i.e. orthogonal projections). These notions are important for functions, too. For instance, given a (complicated) function, we want to find the closest function in a subspace of (simple) functions. In other words, we want to approximate functions using other (typically, simpler) functions.

Comment. Functions $f(x)$ and $g(x)$ can also be multiplied. This is an extra structure (it makes appropriate sets of functions an **algebra**, which is something more special than a **vector space**), which we ignore during our discussion of vector spaces.

Example 170. Which of the following sets are vector spaces?

(a) The set of all functions $\mathbb{R} \to \mathbb{R}$.

(b) The set of all functions $f: \mathbb{R} \to \mathbb{R}$ such that $f(1) = 0$.

(c) The set of all functions $f: \mathbb{R} \to \mathbb{R}$ such that $f(0) = 1$.

(d) The set of all functions $f: \mathbb{R} \to \mathbb{R}$ such that f is differentiable.

Armin Straub
straub@southalabama.edu

Solution.

(a) This set is a vector space.

(b) This set, let's call it Y, is a vector space.

Why? We need to check that it is closed under addition and scaling. If $f(x)$ is a function in Y (that is, $f(1)=0$), then $g(x)=7f(x)$ is also in Y because $g(1)=7f(1)=0$ (of course, 7 can be replaced with any multiple). Similarly, If $f(x)$ and $g(x)$ are functions in Y (that is, $f(1)=0$ and $g(1)=0$), then the sum $h(x)=f(x)+g(x)$ is also in Y because $h(1)=f(1)+g(1)=0+0=0$.

(c) This set is not a vector space.

For instance, it contains the function $f(x)=x+1$ (or, simpler, $f(x)=1$) but does not contain its multiples such as $2f(x)$.

Actually, an easier and more immediate give-away is that the set does not contain the zero "vector" (that is, the function $f(x)=0$, which is everywhere zero).

Important observation. Every vector space must contain a zero vector.

(d) This set is a vector space. Why?!

Example 171. Which of the following sets are vector spaces? For those that are vector spaces, what is the dimension?

(a) The set of all polynomials (with, say, real coefficients).

(b) The set of all polynomials $p(x)$ such that $p(1)=0$.

(c) The set of all polynomials $p(x)$ such that $p(0)=1$.

(d) The set of all polynomials of degree (exactly) 2.

(e) The set of all polynomials of degree 2 or less.

(f) The set of all polynomials $p(x)$ of degree 2 or less such that $p(3)=0$.

Solution.

(a) This set is a vector space of dimension ∞.

Each element of the space is of the form $p(x)=a_0+a_1x+a_2x^2+\ldots+a_nx^n$ for some n. If $a_n\neq 0$, then n is the degree of the polynomial $p(x)$.

(b) As in the previous problem, this set is a vector space of dimension ∞.

(c) As in the previous problem, this set is not a vector space.

(d) This set is not a vector space.

Why? It is not closed under addition: for instance, the sum of $1+2x+x^2$ and $7+3x-x^2$ is not in the set (because it has degree 1).

(e) This set is a vector space of dimension 3.

(We'll look at a basis next time.)

(f) This set is a vector space of dimension $3-1=2$.

(We'll look at a basis next time.)