Example 161. $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ sends $\begin{bmatrix} x \\ y \end{bmatrix}$ to $J\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$. What is the geometric description of this linear map? What are the eigenvalues and eigenvectors?

Solution. Geometrically, this is a rotation by $90°$. This makes it clear that, for no vector $\boldsymbol{x}$ in $\mathbb{R}^2$, we will have $J\boldsymbol{x} = \lambda\boldsymbol{x}$; in fact, $J\boldsymbol{x}$ and $\boldsymbol{x}$ will always be orthogonal!

In other words, J does not have any real (!!) eigenvectors.

Let's go through the math to find complex eigenstuff:

The characteristic polynomial is $\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$, and so A has eigenvalues $\pm i$.

The i-eigenspace is $\text{null}\left(\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\right)$ has basis $\begin{bmatrix} i \\ 1 \end{bmatrix}$. (Indeed, $J\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i\begin{bmatrix} i \\ 1 \end{bmatrix}$.)

The $-i$-eigenspace is $\text{null}\left(\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\right)$ has basis $\begin{bmatrix} -i \\ 1 \end{bmatrix}$. (The exact same, with all i replaced with $-i$.)

Important comment. Since we cannot algebraically distinguish between $\pm i$, we also cannot distinguish between z and $\bar{z}$. This explains that, if we start with a real problem, complex quantities always show up together with their conjugate.

For instance, in this example we saw that $\begin{bmatrix} i \\ 1 \end{bmatrix}$ is a i-eigenvector of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Replacing all i's by $-i$'s, we get that $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ is a $-i$-eigenvector of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. As we knew already.

> Let A be a real matrix. If $\boldsymbol{v}$ is a λ-eigenvector, then $\bar{\boldsymbol{v}}$ is a $\bar{\lambda}$-eigenvector.

See, for instance, Example 161. This is just a consequence of the fact that we cannot algebraically distinguish between $+i$ and $-i$.

Example 162. Show that a symmetric real matrix A must have real eigenvalues.

This statement is part of the spectral theorem.

Solution. Suppose λ is a nonreal eigenvalue with nonzero eigenvector $\boldsymbol{v}$. Then, $\bar{\boldsymbol{v}}$ is a $\bar{\lambda}$-eigenvector and, since $\lambda \neq \bar{\lambda}$, we have two eigenvectors with different eigenvalues. Our computation in Example 109, shows that these two eigenvectors must be orthogonal in the sense that $\bar{\boldsymbol{v}}^T\boldsymbol{v} = 0$. But $\bar{\boldsymbol{v}}^T\boldsymbol{v} = \boldsymbol{v}^*\boldsymbol{v} = \|\boldsymbol{v}\|^2 \neq 0$. This shows that it is impossible to have an nonzero eigenvector for a nonreal eigenvalue.

Euler's formula

Recall that a point (x, y) can be represented using **polar coordinates** (r, θ), where r is the distance to the origin and θ is the angle with the x-axis.

Then, $x = r\cos\theta$ and $y = r\sin\theta$.

> Every complex number z can be written in **polar form** as $z = re^{i\theta}$, with $r = |z|$.

Why? By comparing with the usual polar coordinates ($x = r\cos\theta$ and $y = r\sin\theta$), it only makes sense to write $z = x + iy$ as $z = re^{i\theta}$ if $re^{i\theta} = r\cos\theta + ir\sin\theta$. This is Euler's identity:

Theorem 163. (Euler's identity) $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

- Write down both sides of Euler's identity for $\theta = 0$, $\theta = \frac{\pi}{2}$ and $\theta = \pi$.
  In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects all five fundamental constants).
- Realize that the complex number $\cos(\theta) + i\sin(\theta)$ corresponds to the point $(\cos(\theta), \sin(\theta))$.
  These are precisely the points on the unit circle!
- How can we make sense of the $e^{i\theta}$ in Euler's identity? Using the Taylor series, just as we did for e^A:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix - \frac{x^2}{2} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \ldots$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots \qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Example 164. Where do trig identities like $\sin(2x) = 2\cos(x)\sin(x)$ or $\sin^2(x) = \frac{1-\cos(2x)}{2}$ (and infinitely many others you have never heard of!) come from?
Short answer: they all come from the simple exponential law $e^{x+y} = e^x e^y$.

Let us illustrate this in the simple case $(e^x)^2 = e^{2x}$. Observe that

$$\begin{aligned} e^{2ix} &= \cos(2x) + i\sin(2x) \\ e^{ix}e^{ix} &= [\cos(x) + i\sin(x)]^2 = \cos^2(x) - \sin^2(x) + 2i\cos(x)\sin(x). \end{aligned}$$

Comparing imaginary parts (the "stuff with an i"), we conclude that $\sin(2x) = 2\cos(x)\sin(x)$.
Likewise, comparing real parts, we read off $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(Use $\cos^2(x) + \sin^2(x) = 1$ to derive $\sin^2(x) = \frac{1-\cos(2x)}{2}$ from the last equation.)

Challenge. Can you find a triple-angle trig identity for $\cos(3x)$ and $\sin(3x)$ using $(e^x)^3 = e^{3x}$?
Or, use $e^{i(x+y)} = e^{ix}e^{iy}$ to derive $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x+y) = \ldots$ (that's what we actually did in class).

Example 165. Solve the differential equation

$$\boldsymbol{y}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solution. From Example 161, we know that $A = PDP^{-1}$ with $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.
The system is therefore solved by:

$$\begin{aligned} \boldsymbol{y}(t) &= Pe^{Dt}P^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{it} & \\ & e^{-it} \end{bmatrix}\frac{1}{2i}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{2i}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{it} & \\ & e^{-it} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{it} \\ -e^{-it} \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} ie^{it} + ie^{-it} \\ e^{it} - e^{-it} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} e^{it} + e^{-it} \\ -ie^{it} + ie^{-it} \end{bmatrix} \end{aligned}$$

Using Euler's theorem, we can rewrite this solution in terms of $\cos(t)$ and $\sin(t)$. For instance, $e^{it} + e^{-it} = (\cos(t) + i\sin(t)) + (\cos(-t) + \sin(-t)) = 2\cos(t)$.

In conclusion, $\boldsymbol{y}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$. (Check by plugging into the differential equation!)

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