Review.

- The solution to the differential equation

$$\boldsymbol{y}' = \underbrace{\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}}_{A}\boldsymbol{y}, \qquad \boldsymbol{y}(0) = \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\boldsymbol{y}_0}$$

is

$$\begin{aligned}
\boldsymbol{y}(t) &= e^{At}\boldsymbol{y}_0 \\
&= e^{2It+Nt}\boldsymbol{y}_0 \qquad \text{with } N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= e^{2It}e^{Nt}\boldsymbol{y}_0 \qquad \text{(because } 2It \text{ and } Nt \text{ commute)} \\
&= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\left(1 + Nt + \frac{1}{2}(Nt)^2 + \frac{1}{3!}(Nt)^3 + \ldots\right)\boldsymbol{y}_0 \\
&= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}(1 + Nt)\boldsymbol{y}_0 \qquad \text{(because } N^2 = \mathbf{0}) \\
&= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix}\begin{bmatrix} t-1 \\ 1 \end{bmatrix} = \begin{bmatrix} (t-1)e^{2t} \\ e^{2t} \end{bmatrix}.
\end{aligned}$$

Important comment. Note that we can immediately see from the solution that the original matrix A is not diagonalizable: there is a term te^{2t}, whereas in the diagonalizable case we would only see exponentials like e^{2t} by themselves.

In our upcoming discussion of complex numbers we will see that e^{2it} (here, $2i$ would be the eigenvalue) can be rewritten in terms of $\cos(2t)$ and $\sin(2t)$. Both of these are periodic and bounded, so that the same is true for any linear combination.

In that case, if the eigenvalue $2i$ was repeated in such a way that the matrix A is not diagonalizable, then we would get the functions $t\cos(2t)$ and $t\sin(2t)$ in our solutions. These, however, are not bounded! This phenomenon (getting solutions that are unbounded under the right/wrong circumstances) is called **resonance**.

https://en.wikipedia.org/wiki/Resonance

Understanding when resonance occurs is of crucial importance for pratical applications.

Some comments on complex numbers

Let's recall some very basic facts about **complex numbers**:

- Every complex number can be written as $z = x + iy$ with real x, y.
- Here, the imaginary unit i is characterized by solving $x^2 = -1$.

  Important observation. The same equation is solved by $-i$. This means that, algebraically, we cannot distinguish between $+i$ and $-i$.
- The **conjugate** of $z = x + iy$ is $\bar{z} = x - iy$.

  Important comment. Since we cannot algebraically distinguish between $\pm i$, we also cannot distinguish between z and $\bar{z}$. This explains that, if we start with a real problem, complex quantities always show up together with their conjugate.

Example 159. (warmup) What is $\frac{1}{2 + 3i}$?

Solution. $\frac{1}{2 + 3i} = \frac{2 - 3i}{(2 + 3i)(2 - 3i)} = \frac{2 - 3i}{13}$.

- In general, $\frac{1}{z} = \frac{\bar{z}}{z\,\bar{z}} = \frac{\bar{z}}{|z|^2}$.
- The **absolute value** of the complex number $z = x + iy$ is $|z| = \sqrt{x^2 + y^2} = \sqrt{\bar{z}z}$.
- The **norm** of the complex vector $\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is $\|\boldsymbol{z}\| = \sqrt{|z_1|^2 + |z_2|^2}$.

  Note that $\|\boldsymbol{z}\|^2 = \bar{z}_1 z_1 + \bar{z}_2 z_2 = \bar{\boldsymbol{z}}^T \boldsymbol{z}$.

Definition 160.

- For any matrix A, its **conjugate transpose** is $A^* = (\bar{A})^T$.
- The **dot product** (inner product) of complex vectors is $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \boldsymbol{v}^* \boldsymbol{w}$.
- A complex $n \times n$ matrix A is **unitary** if $A^* A = I$.

Comment. A^* is also written A^H (or $A^\dagger$ in quantum mechanics) and called the Hermitian conjugate.

Comment. For real matrices and vectors, the conjugate transpose is just the ordinary transpose. In particular, the dot product is the same.

Comment. Unitary matrices are the complex version of orthogonal matrices. (A real matrix if unitary if and only if it is orthogonal.) Again, a matrix A is unitary if and only if $\langle A\boldsymbol{v}, A\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

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