**Example 155.** What are the possible Jordan normal forms of a $6\times 6$ matrix with eigenvalues $3,3,7,7,7,7$?

**Solution.** There are $2\cdot 5=10$ possible Jordan normal forms for such a matrix:

$$\begin{bmatrix} 3 &&&&& \\ & 3 &&&& \\ && 7 &&& \\ &&& 7 && \\ &&&& 7 & \\ &&&&& 7 \end{bmatrix}, \begin{bmatrix} 3 &&&&& \\ & 3 &&&& \\ && 7 & 1 && \\ &&& 7 && \\ &&&& 7 & \\ &&&&& 7 \end{bmatrix}, \begin{bmatrix} 3 &&&&& \\ & 3 &&&& \\ && 7 & 1 && \\ &&& 7 && \\ &&&& 7 & 1 \\ &&&&& 7 \end{bmatrix}, \begin{bmatrix} 3 &&&&& \\ & 3 &&&& \\ && 7 & 1 && \\ &&& 7 & 1 & \\ &&&& 7 & \\ &&&&& 7 \end{bmatrix}, \begin{bmatrix} 3 &&&&& \\ & 3 &&&& \\ && 7 & 1 && \\ &&& 7 & 1 & \\ &&&& 7 & 1 \\ &&&&& 7 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 1 &&&& \\ & 3 &&&& \\ && 7 &&& \\ &&& 7 && \\ &&&& 7 & \\ &&&&& 7 \end{bmatrix}, \begin{bmatrix} 3 & 1 &&&& \\ & 3 &&&& \\ && 7 & 1 && \\ &&& 7 && \\ &&&& 7 & \\ &&&&& 7 \end{bmatrix}, \begin{bmatrix} 3 & 1 &&&& \\ & 3 &&&& \\ && 7 & 1 && \\ &&& 7 && \\ &&&& 7 & 1 \\ &&&&& 7 \end{bmatrix}, \begin{bmatrix} 3 & 1 &&&& \\ & 3 &&&& \\ && 7 & 1 && \\ &&& 7 & 1 & \\ &&&& 7 & \\ &&&&& 7 \end{bmatrix}, \begin{bmatrix} 3 & 1 &&&& \\ & 3 &&&& \\ && 7 & 1 && \\ &&& 7 & 1 & \\ &&&& 7 & 1 \\ &&&&& 7 \end{bmatrix}$$

**Example 156.** How many different Jordan normal forms are there in the following cases?

(a) A $8\times 8$ matrix with eigenvalues $1,1,2,2,2,4,4,4$?

(b) A $11\times 11$ matrix with eigenvalues $1,1,1,2,2,2,2,4,4,4,4$?

**Solution.**

(a) $2\cdot 3\cdot 3=18$ possible Jordan normal forms

(b) $3\cdot 5\cdot 5=75$ possible Jordan normal forms

**Review.**

- Let $A$ be $n\times n$. The matrix exponential is

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

  Then, $\frac{\mathrm{d}}{\mathrm{d}t}e^{At} = Ae^{At}$.

  **Why?** $\frac{\mathrm{d}}{\mathrm{d}t}e^{At} = \frac{\mathrm{d}}{\mathrm{d}t}\left(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right) = A + \frac{1}{1!}A^2t + \frac{1}{2!}A^3t^2 + \cdots = Ae^{At}$

- If $A = PDP^{-1}$, then $e^A = Pe^DP^{-1}$.
- The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

  **Why?** Because $\boldsymbol{y}'(t) = Ae^{At}\boldsymbol{y}_0 = A\boldsymbol{y}(t)$ and $\boldsymbol{y}(0) = e^{0A}\boldsymbol{y}_0 = \boldsymbol{y}_0$.

**Example 157.** The matrix exponential shares many other properties of the usual exponential:

- $e^Ae^B = e^{A+B} = e^Be^A$ if $AB = BA$

  **Why the condition $AB = BA$?** By the Taylor series, $e^{A+B} = I + (A+B) + \frac{(A+B)^2}{2!} + \ldots$ In order to simplify that to

$$e^Ae^B = \left(I + A + \frac{A^2}{2!} + \ldots\right)\left(I + B + \frac{B^2}{2!} + \ldots\right),$$

  we need that $(A+B)^2 = A^2 + AB + BA + B^2$ is the same as $A^2 + 2AB + B^2$. That's only the case if $AB = BA$.

- $e^A$ is invertible and $(e^A)^{-1} = e^{-A}$

  **Why?** That actually follows from the previous property.

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**Example 158.** Solve the differential equation

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

**Solution.**

- If $A = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}$, then the solution is $\boldsymbol{y}(t) = e^{At} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

  The only difficulty is in computing $e^{At}$ since we already observed that $A$ is not diagonalizable.

- Write $A = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} = 2I + N$ with $N = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}$. Note that $2I$ and $N$ commute.

  Hence, $e^{At} = e^{2It + Nt} = e^{2It} e^{Nt}$.

- Note that $N^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, $e^{Nt} = I + Nt + \frac{t^2}{2!} N^2 + \ldots = I + Nt = \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$.

- Combined, $e^{At} = e^{2It + Nt} = e^{2It} e^{Nt} = \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & t e^{2t} \\ & e^{2t} \end{bmatrix}$.

  In particular, $\boldsymbol{y}(t) = e^{At} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} (t-1)\, e^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} t-1 \\ 1 \end{bmatrix} e^{2t}$.

**Check.** We should verify that $y_1 = (t-1)e^{2t}$ and $y_2 = e^{2t}$ satisfy $y_1' = 2y_1 + y_2$ and $y_2' = 2y_2$.
Indeed, $y_1' = e^{2t} + (t-1)2e^{2t}$ equals $2y_1 + y_2 = 2(t-1)e^{2t} + e^{2t}$.

**Comment.** For applications, having solutions like $te^{\lambda t}$ or $t\cos(\lambda t)$ (when the eigenvalues are imaginary) is connected to the phenomenon of **resonance**, which you may have already seen.

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