**Example 111.** Determine the SVD of $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.

**Solution.** $A^T A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ has $8$-eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $2$-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Since $A^T A = V \Sigma^2 V^T$ (here, $\Sigma^T \Sigma = \Sigma^2$), we conclude that $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{8} & \\ & \sqrt{2} \end{bmatrix}$.

From $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$, we find $\boldsymbol{u}_1 = \frac{1}{\sigma_1} A \boldsymbol{v}_1 = \frac{1}{\sqrt{8}} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Likewise, $\boldsymbol{u}_2 = \frac{1}{\sigma_2} A \boldsymbol{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence, $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Check that, indeed, $A = U \Sigma V^T$!

**Comment.** For applications, it is common to arrange the singular values in decreasing order like we did.

**Comment.** If we had chosen $V = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$ instead, then $U = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{8} & \\ & \sqrt{2} \end{bmatrix}$.

As with diagonalization, there is choices! This is a perfectly fine SVD. In fact, it's what Sage computes below.

**Sage.** Let's have Sage do the work for us. In Sage, the SVD is currently only implemented for floating point numbers. (RDF is the real numbers as floating point numbers with double precision)

```
Sage] A = matrix(RDF, [[2,2],[-1,1]])
```

```
Sage] U,S,V = A.SVD()
```

```
Sage] U
```

$$\begin{bmatrix} -1.0 & 1.11022302463 \times 10^{-16} \\ 8.64109131471 \times 10^{-17} & 1.0 \end{bmatrix}$$

```
Sage] S
```

$$\begin{bmatrix} 2.82842712475 & 0.0 \\ 0.0 & 1.41421356237 \end{bmatrix}$$

```
Sage] V
```

$$\begin{bmatrix} -0.707106781187 & -0.707106781187 \\ -0.707106781187 & 0.707106781187 \end{bmatrix}$$

**Example 112.** Determine the SVD of $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$.

**Comment.** In contrast to our previous example, $\operatorname{rank}(A) = 1$. It follows that $A^T A$ has eigenvalue $0$, so that $0$ is a singular value of $A$.

**Solution.** $A^T A = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$ has $10$-eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $0$-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

We conclude that $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$.

$\boldsymbol{u}_1 = \frac{1}{\sigma_1} A \boldsymbol{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{20}} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

We cannot obtain $\boldsymbol{u}_2$ in the same way because $\sigma_2 = 0$. Since for every vector $\boldsymbol{u}_2$, $A\boldsymbol{v}_2 = \sigma_2 \boldsymbol{u}_2$, we can choose $\boldsymbol{u}_2$ as we wish, as long as the columns of $U$ are orthonormal in the end.

$\boldsymbol{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ (but $\boldsymbol{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ works just as well)

Hence, $U = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

In summary, $A = U \Sigma V^T$ with $U = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

**Check.** Do check that, indeed, $A = U \Sigma V^T$.

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