More details on the spectral theorem

Let us add $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ to our notations for the dot product: $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \boldsymbol{v}^T \boldsymbol{w} = \boldsymbol{v} \cdot \boldsymbol{w}$.

- In our story of orthogonality, the important player has been the dot product. However, one could argue that the fundamental quantity is actually the norm:
  $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \frac{1}{4}(\|\boldsymbol{v}+\boldsymbol{w}\|^2 - \|\boldsymbol{v}-\boldsymbol{w}\|^2)$. See Example 19.
- Accepting the dot product as immensely important, we see that symmetric matrices (i.e. matrices A such that $A = A^T$) are of interest.
  For any matrix A, $\langle A\boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{v}, A^T \boldsymbol{w} \rangle$.
  It follows that, a matrix A is symmetric if and only if $\langle A\boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{v}, A\boldsymbol{w} \rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.
- Similarly, let Q be an orthogonal matrix (i.e. Q is a square matrix with $Q^T Q = I$).
  Then, $\langle Q\boldsymbol{v}, Q\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \boldsymbol{w} \rangle$.
  In fact, a matrix A is orthogonal if and only if $\langle A\boldsymbol{v}, A\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.
  Comment. We observed in Example 77 that orthogonal matrices Q correspond to rotations ($\det Q = 1$) or reflections ($\det Q = -1$) [or products thereof]. The equality $\langle Q\boldsymbol{v}, Q\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ encodes the fact that these types (and only these!) of geometric transformations preserve angles and lengths.

(Spectral theorem)

Every symmetric $n \times n$ matrix A can be decomposed as $A = PDP^T$, where

- D is a diagonal matrix, $(n \times n)$

  The diagonal entries λ_i are the **eigenvalues** of A.

- P is orthogonal. $(n \times n)$

  The columns of P are **eigenvectors** of A.

Note that, in particular, A is always diagonalizable, the eigenvalues (and hence, the eigenvectors) are all real, and, most importantly, the eigenspaces of A are orthogonal.

Let us prove that, indeed, the eigenspaces of a symmetric matrix are orthogonal:

Example 109. Suppose A is symmetric. Show that the eigenspaces of A are orthogonal.

Solution. We need to show that, if $\boldsymbol{v}$ and $\boldsymbol{w}$ are eigenvectors of A with different eigenvalues, then $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \boldsymbol{0}$.
Suppose that $A\boldsymbol{v} = \lambda \boldsymbol{v}$ and $A\boldsymbol{w} = \mu \boldsymbol{w}$ with $\lambda \neq \mu$.
Then, $\lambda \langle \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \lambda \boldsymbol{v}, \boldsymbol{w} \rangle = \langle A\boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{v}, A^T \boldsymbol{w} \rangle = \langle \boldsymbol{v}, A\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \mu \boldsymbol{w} \rangle = \mu \langle \boldsymbol{v}, \boldsymbol{w} \rangle$.
However, since $\lambda \neq \mu$, $\lambda \langle \boldsymbol{v}, \boldsymbol{w} \rangle = \mu \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ is only possible if $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$.

Example 110. By the spectral theorem, every symmetric matrix A can be written as $A = VDV^T$ for a diagonal matrix D and an orthogonal matrix V. What about A^{-1}?

Solution. Recall that $(AB)^{-1} = B^{-1}A^{-1}$, for any two invertible matrices A, B.
If $A = VDV^T$, then $A^{-1} = (V^T)^{-1} D^{-1} V^{-1}$. Since $V^{-1} = V^T$, this simplifies to $A^{-1} = VD^{-1}V^T$.
Comment. Likewise, $A^n = VD^nV^T$.

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Singular value decomposition

(Singular value decomposition)

Every $m \times n$ matrix A can be decomposed as $A = U\Sigma V^T$, where

- Σ is a (rectangular) diagonal matrix with nonnegative entries, $(m \times n)$

The diagonal entries σ_i are called the **singular values** of A.

- U is orthogonal, $(m \times m)$
- V is orthogonal. $(n \times n)$

Comment. If A is symmetric, then the singular value decomposition is already provided by the spectral theorem (the diagonalization of A). Moreover, in that case, $V = U$.

Important observations. If $A = U\Sigma V^T$, then $A^T A = V\Sigma^T \Sigma V^T$.

- Note that $\Sigma^T \Sigma$ is an $n \times n$ diagonal matrix. Its entries are σ_i^2 (the squares of the entries in Σ).
- $A^T A$ is a symmetric matrix! (Why?!) Hence, by the spectral theorem, we are able to find V and $\Sigma^T \Sigma$.

In other words, V is obtained from the (orthonormally chosen) eigenvectors of $A^T A$. Likewise, the entries of $\Sigma^T \Sigma$ are the eigenvalues of $A^T A$; their square roots are the entries of Σ, the singular values.

Finally, the equation $AV = U\Sigma$ allows us to determine U. How?! (Hint: $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$)

This results in the following **recipe** to determine the SVD $A = U\Sigma V^T$ for any matrix A.

Find an orthonormal basis of eigenvectors $\boldsymbol{v}_i$ of $A^T A$. Let λ_i be the eigenvalue of $\boldsymbol{v}_i$.

- V is the matrix with columns $\boldsymbol{v}_i$.
- Σ is the diagonal matrix with entries $\sigma_i = \sqrt{\lambda_i}$.
- U is the matrix with columns $\boldsymbol{u}_i = \frac{1}{\sigma_i} A\boldsymbol{v}_i$. If needed, fill in additional columns to make U orthogonal.

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