Example 102. Which of the following are true for all square matrices A?

- Is it true that A^T has the same eigenvalues as A?
- Is it true that A^T has the same eigenspaces as A?
- Is it true that A^T has the same characteristic polynomial as A?

Solution. True. False. True.

First, note that the characteristic polynomial $\det(A-\lambda I)$ is the same as $\det(A^T-\lambda I)$. [Make sure you can fill in the details of why this is the case!] Hence, the eigenvalues (which are the roots of the characteristic polynomial) are also the same for A and A^T.

On the other hand, A^T and A in general have very different eigenspaces. Take, for instance, the matrix $A=\begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$ from Example 100. Then both A and A^T have eigenvalues $\lambda=0.5, 1$.

However, the 1-eigenspace of A is spanned by $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, while the 1-eigenspace of A^T is spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

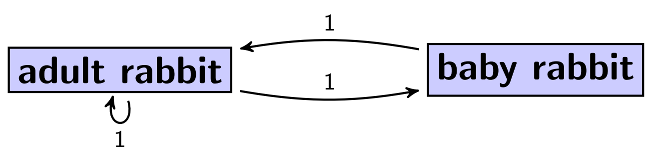
Example 103. Show that a Markov matrix A (so that the columns of A sum to 1) always has eigenvalue 1.

Solution. This follows because the transpose A^T always has $[\,1 \;\; 1 \;\; \ldots \;\; 1\,]^T$ as an 1-eigenvector (by virtue of the rows of A^T summing to 1). [Make sure that makes sense!]

By the previous example, A must also have eigenvalue 1 (but we have no idea what a 1-eigenvector is until we compute it).

Example 104. We model rabbit reproduction as follows.

Each month, every pair of adult rabbits produces one pair of baby rabbit as offspring. Meanwhile, it takes baby rabbits one month to mature to adults.



Comment. In this simplified model, rabbits always come in male/female pairs and no rabbits die. Though these feautures might make it sound completely useless, the model may have some merit when describing populations under ideal conditions (unlimited resources) and over short time (no deaths).

Historical comment. The question how many rabbits there are after one year, when starting out with a pair of baby rabbits is famously included in the 1202 textbook of the Italian mathematician Leonardo of Pisa, known as Fibonacci.

Describe the transition from one month to the next.

Solution. Let x_t be the number of adult rabbit pairs after t months. Likewise, y_t is the number of baby rabbit pairs. Then the transition from one month to the next is described by

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} x_t + y_t \\ x_y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ y_y \end{bmatrix}.$$

Determine several powers of $T=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and interpret the values in each column of T^n.

Solution. $T^2=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $T^3=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$, $T^4=\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$,
$T^5=\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 8 & 5 \\ 5 & 3 \end{bmatrix}$, $T^6=\begin{bmatrix} 8 & 5 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix}$

The first column of T^n equals $T^n\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the state of 1 adult rabbit pair and 0 baby rabbits. Hence, $T^n\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} a \\ b \end{bmatrix}$ where a (respectively, b) is the number of adult (respectively, baby) rabbit pairs after n months. (Check that this matches the values we obtained in the first column of $T^2, \ldots, T^6$.)

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You probably recognize the numbers we are getting: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

These are **Fibonacci numbers**! How fast are they growing?

Did you notice that $\frac{2}{1} = 2$, $\frac{3}{2} = 1.5$, $\frac{5}{3} = 1.6$, $\frac{13}{8} = 1.625$, $\frac{21}{13} = 1.615$, $\frac{34}{21} = 1.619$, ...

These ratios approach the **golden ratio** $\varphi = 1.618\ldots$ Where's that coming from?

- Let us write F_n for the n-th Fibonacci number, starting with $F_0 = 0$, $F_1 = 1$. Our earlier observation translates into $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$ and, thus, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

- The eigenvalues of $T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ are $\lambda_1 = \frac{1 + \sqrt{5}}{2} \approx 1.618$ (the golden ratio!) and $\lambda_2 = \frac{1 - \sqrt{5}}{2} \approx -0.618$.

- The corresponding eigenvectors are $\boldsymbol{v}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$.

- In terms of the basis of eigenvectors, we have $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2$ with $c_1 = \frac{1}{\sqrt{5}}$, $c_2 = -\frac{1}{\sqrt{5}}$.

- Hence, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = T^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = T^n (c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2) = \lambda_1^n c_1 \boldsymbol{v}_1 + \lambda_2^n c_2 \boldsymbol{v}_2$.

  In particular, focusing on the second entry,

  $$F_n = \lambda_1^n c_1 + \lambda_2^n c_2 = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right].$$

  That's **Binet's formula**.

- For large n, $F_n \approx \lambda_1^n c_1$ (because λ_2^n becomes very small).

  In particular, it is now transparent that the ratios $\frac{F_{n+1}}{F_n}$ approach $\lambda_1 = \frac{1 + \sqrt{5}}{2} \approx 1.618$.

Comment. In fact, since λ_2 is so small, $F_n = \mathrm{round}\left(\frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n \right)$.

Advanced comment. Note that the transition matrix connected to the Fibonacci numbers can be obtained directly from the recursive relation $F_{n+1} = F_n + F_{n-1}$, with which the Fibonacci numbers are usually introduced. That's because the recursion is equivalent to $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$.

More importantly, we see that, given any such recursion, we can likewise apply our linear algebra skills.
[Recursions are a discrete analog of differential equations.]

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