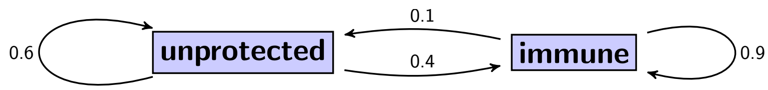
Sketch of Lecture 18

Application: Markov chains

Example 100. Consider a fixed population of people with or without active immunization against some disease (like tetanus). Suppose that, each year, 40% of those unprotected get vaccinated while 10% of those with immunization lose their protection.

What is the immunization rate in the long run? (The long term equilibrium.)

Solution.



x_t: proportion of population unprotected at time t (in years)

y_t: proportion of population immune at time t [Note that $x_t + y_t = 1$.]

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0.6x_t + 0.1y_t \\ 0.4x_t + 0.9y_t \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

The matrix $\begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$ is the **transition matrix** of this dynamical system, because it describes the transition from time t to time $t+1$. This particular one is a **Markov matrix** (or stochastic matrix): its columns add to 1 and it has no negative entries.

$\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an equilibrium if $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$. In other words, $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an eigenvector with eigenvalue 1.

The 1-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -0.4 & 0.1 \\ 0.4 & -0.1 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Since $x_\infty + y_\infty = 1$, we conclude that $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \frac{1}{1+4}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 4/5 \end{bmatrix}$.

Hence, the immunization rate in the long term equilibrium is $4/5 = 80\%$.

[Ponder about why this is a reasonable value!]

Comment. What's the other eigenvalue of the transition matrix? No need to compute the characteristic polynomial: we can easily see that it is $0.5 = 0.6 \cdot 0.9 - 0.1 \cdot 0.4$ because the product of the eigenvalues equals the determinant!

The 0.5-eigenspace is spanned by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

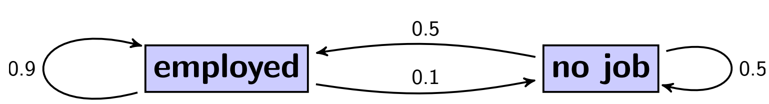
Comment. Will the immunization rate always stabilize and approach the long term equilibrium? Yes! This is a consequence of the other eigenvalue of the transition matrix satisfying $|0.5| < 1$. If we start in state $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 4 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = 1^n \cdot a\begin{bmatrix} 1 \\ 4 \end{bmatrix} + (0.5)^n \cdot b\begin{bmatrix} -1 \\ 1 \end{bmatrix} \underset{\text{as } n\to\infty}{\longrightarrow} a\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Random comment. A rule of thumb is that tetanus vaccination begins to wear off after about 10 years (somewhat in line with the 0.1 transition proportion in this example). However, the tetanus immunzation rate in the United States appears to be considerable less than the 80% we found in this (awfully simplistic) example.

https://www.cdc.gov/mmwr/preview/mmwrhtml/mm5940a3.htm

Example 101. (extra) Consider a fixed population of people with or without a job. Suppose that, each year, 50% of those unemployed find a job while 10% of those employed lose their job. What is the unemployment rate in the long term equilibrium?

Solution.



Proceeding, as in the previous example, the transition matrix is $\begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}$.

In the end, we find that the unemployment rate in the long term equilibrium is $1/6 \approx 16.7\%$.

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