Theorem 86. (spectral theorem, compact version) A symmetric matrix A can always be diagonalized as $A = PDP^T$, where P is orthogonal and D is diagonal (and both are real).

How? We proceed as in the diagonalization $A = PDP^{-1}$. We then arrange P to be orthogonal, by normalizing its columns. If there is a repeated eigenvalue, then we also need to make sure to pick an orthonormal basis for the corresponding eigenspace (for instance, using Gram–Schmidt).

Advanced comment. A matrix such that $A^T A = AA^T$ is called **normal**. In a similar spirit as in Example 94 one can show that, for normal matrices, the eigenspaces are orthogonal to each other. However, normal matrices which are not symmetric will always have complex eigenvalues. (In that case, the orthogonal matrix P gets replaced with a unitary matrix, the complex version of orthogonal matrices, and the P^T becomes the conjugate transpose $P^* = \bar{P}^T$.)

Example 87. (warmup) What are the eigenvalues and eigenvectors of the 3×3 identity?

Solution. The eigenvalues are $1, 1, 1$. The 1-eigenspace is all of $\mathbb{R}^3$. (In other words, every vector is a 1-eigenvector.)

Example 88. (warmup) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ sends $\begin{bmatrix} x \\ y \end{bmatrix}$ to $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$. What is the geometric description of this linear map? What are the eigenvalues and eigenvectors?
Diagonalize A as PDP^T.

Solution. Geometrically, this is a reflection through the line $y = x$. Make a sketch!

This description makes it obvious that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a 1-eigenvector, and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a -1-eigenvector.

(Of course, we can also just compute these. But do make sure that this is obvious geometrically.)

To get an orthogonal matrix P, we normalize $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ to $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Hence, for the diagonalization of A as $A = PDP^T$, we can choose $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Important comment. For the diagonalization of A as $A = PDP^{-1}$, we can just choose $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. However, P is not orthogonal and so $P^{-1} \neq P^T$ (in fact, $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}P^T$). That's why we need to normalize the columns of P for the diagonalization $A = PDP^T$.

Comment. Note that the language of eigenthings makes it easy to identify and construct reflections (and other geometric transformations). See next example.

Comment. Note that the determinant of A is -1. Areas are preserved but the orientation is changed.

Example 89. (extra) Diagonalize the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & -7 \end{bmatrix}$ as $A = PDP^T$.

Solution. The characteristic polynomial is $\begin{vmatrix} 1-\lambda & 3 \\ 3 & -7-\lambda \end{vmatrix} = (1-\lambda)(-7-\lambda) - 9 = (\lambda+8)(\lambda-2)$, and so A has eigenvalues $-8, 2$.

The 2-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

The -8-eigenspace is $\operatorname{null}\left(\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Hence, if $P = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & -8 \end{bmatrix}$, then $A = PDP^T$.

Important observation. The 2-eigenvector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and the -8-eigenvector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ are indeed orthogonal!

We normalize the two eigenvectors to $\frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. These are an orthonormal basis for $\mathbb{R}^2$.

Comment (again). Note that we were asked for a diagonalization of the form $A = PDP^T$. For that, the matrix P must be orthogonal. In particular, we must normalize its columns! (Otherwise, we only have $A = PDP^{-1}$.)

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Example 90. (review) If A is a 2×2 matrix with $\det(A) = -8$ and eigenvalue 4. What is the second eigenvalue?

Solution. Recall that $\det(A)$ is the product of the eigenvalues (see below). Hence, the second eigenvalue is -2.

$\det(A)$ is the product of the eigenvalues of A.

Why? Recall how we determine the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an $n \times n$ matrix A. We compute the characteristic polynomial $\det(A - \lambda I)$ and determine the λ_i as the roots of that polynomial.

That means that we have the factorization $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_n - \lambda)\cdots(\lambda_n - \lambda)$. Now, set $\lambda = 0$ to conclude that $\det(A) = \lambda_1\lambda_2\cdots\lambda_n$.

Example 91. Diagonalize the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ as $A = PDP^T$.

Solution. The characteristic polynomial is $\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (\lambda - 4)(\lambda + 2)$, and so A has eigenvalues $4, -2$.

The 4-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The -2-eigenspace is $\operatorname{null}\left(\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

In summary, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a 4-eigenvector, and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a -2-eigenvector.

Review. The product of all eigenvalues $-2\cdot 4 = -8$ always equals the determinant $\det(A) = 1 - 9 = -8$.

Example 92. Sketch the effect of $\boldsymbol{x} \mapsto A\boldsymbol{x}$ with $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ in the following two ways:

(a) Where are the standard basis vectors $\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ being sent? Also sketch where the square spanned by these two vectors is sent.

(b) Repeat using the orthonormal basis $\boldsymbol{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Solution. See blackboard. Of course, $A\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $A\boldsymbol{e}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. This means that the square spanned by $\boldsymbol{e}_1, \boldsymbol{e}_2$ (a square) is sent to the parallelogram spanned by $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Moreover, if we keep track of the sides, we see that the parallelogram is flipped.

In the second case, the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ just get stretched (by a factor of 4 and -2, respectively). In particular, the square spanned by $\boldsymbol{v}_1, \boldsymbol{v}_2$ is sent to a rectangle.

Important comment. The second sketch makes the geometric interpretation of the determinant ($\det(A) = -8$) plainly visible. Namely, areas get increased by a factor of 8 (the 1×1 square is mapped to a 4×2 rectangle). The negative sign indicates that the square also gets flipped.

Example 93. (extra) Diagonalize the symmetric matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ as $A = PDP^T$.

Solution. (final solution only)

$A = PDP^T$ with $P = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} \end{bmatrix}$ and $D = \begin{bmatrix} 1+2\sqrt{2} & & \\ & 1-2\sqrt{2} & \\ & & 1 \end{bmatrix}$

Comment. Note that we were asked for a diagonalization of the form $A = PDP^T$. For that, the matrix P must be orthogonal. In particular, we must normalize its columns! (Otherwise, we only have $A = PDP^{-1}$.)

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