Example 71. Determine the QR decomposition of $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$.

Solution. We first apply Gram–Schmidt orthonormalization to the columns of A.

- $\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so that $\boldsymbol{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.
- $\boldsymbol{b}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \cdot \boldsymbol{q}_1 \right) \boldsymbol{q}_1 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$, so that $\boldsymbol{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
- $\boldsymbol{b}_3 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \left(\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \cdot \boldsymbol{q}_1 \right) \boldsymbol{q}_1 - \left(\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \cdot \boldsymbol{q}_2 \right) \boldsymbol{q}_2 = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$, so that $\boldsymbol{q}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Therefore, $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Finally, $R = Q^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.

In conclusion, we have found the QR decomposition: $\underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}}_{R}$

Example 72. (review) A matrix A has orthonormal columns $\iff$ $A^T A = I$.

Definition 73. An **orthogonal matrix** is a square matrix with orthonormal columns.

[This is not a typo (but a confusing convention): the columns need to orthonormal, not just orthogonal.]

An $n \times n$ matrix Q is orthogonal $\iff$ $Q^T Q = I$

In other words, $Q^{-1} = Q^T$.

Example 74. Suppose A is invertible. What is the projection matrix P for orthogonally projecting onto $\operatorname{col}(A)$?

Solution. If A is an invertible $n \times n$ matrix, then $\operatorname{col}(A) = \mathbb{R}^n$ (because the n columns of A are linearly independent and hence form a basis for $\mathbb{R}^n$).

Since $\operatorname{col}(A)$ is the entire space we are not really projecting at all: every vector is sent to itself.

In particular, the projection matrix is $P = I$.

Example 75. Suppose Q has orthonormal columns. What is the projection matrix P for orthogonally projecting onto $\operatorname{col}(Q)$?

Solution. Recall that, to project onto $\operatorname{Col}(A)$, the projection matrix is $P = A(A^T A)^{-1} A^T$.

Since $Q^T Q = I$, to project onto $\operatorname{Col}(Q)$, the projection matrix is $P = Q Q^T$.

Comment. A familiar special case is when we project onto a unit vector $\boldsymbol{q}$: in that case, the projection of $\boldsymbol{b}$ onto $\boldsymbol{q}$ is $(\boldsymbol{q} \cdot \boldsymbol{b}) \boldsymbol{q} = \boldsymbol{q}(\boldsymbol{q}^T \boldsymbol{b}) = (\boldsymbol{q}\boldsymbol{q}^T)\boldsymbol{b}$, so the projection matrix here is $\boldsymbol{q}\boldsymbol{q}^T$.

Comment. In particular, if Q is not square, then $Q^T Q = I$ but $Q Q^T \neq I$. In some sense, $Q Q^T$ still "tries" to be as close to the identity as possible: since it is the matrix projecting onto $\operatorname{col}(Q)$ it does act like the identity for vectors in $\operatorname{col}(Q)$. (Vectors not in $\operatorname{col}(Q)$ are sent to their projection, that is, the closest to themselves while restricted to $\operatorname{col}(Q)$.)