Example 55. If P is a projection matrix, then what is P^2?

For instance. For P as in Example 52, $P^2 = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P$.

Solution. Can you see why it is always true that $P^2 = P$?

[Recall that P projects a vector onto a space W (actually, $W = \mathrm{col}(P)$). Hence P^2 takes a vector $\boldsymbol{b}$, projects it onto W to get $\hat{\boldsymbol{b}}$, and then projects $\hat{\boldsymbol{b}}$ onto W again. But the projection of $\hat{\boldsymbol{b}}$ onto W is just $\hat{\boldsymbol{b}}$ (why?!), so that P^2 always has the exact same effect as P. Therefore, $P^2 = P$.]

Example 56. True or false? If P is the matrix for projecting onto W, then $W = \mathrm{col}(P)$.

Solution. True!

Why? The columns of P are the projections of the standard basis vectors and hence in W. On the other hand, for any vector $\boldsymbol{w}$ in W, we have $P\boldsymbol{w} = \boldsymbol{w}$ so that $\boldsymbol{w}$ is a combination of the columns of P.

[This may take several readings to digest but do read (or ask) until it makes sense!]

In particular. $\mathrm{rank}(P) = \dim W$ (because, for any matrix, $\mathrm{rank}(A) = \dim \mathrm{col}(A)$)

Review.

- Vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are **linearly independent**.

  $\Longleftrightarrow$ $c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n = \boldsymbol{0}$ only has the (trivial) solution $c_1 = c_2 = \ldots = c_n = 0$.

- Vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are a **basis** for V.

  $\Longleftrightarrow$ $V = \mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent.

  $\Longleftrightarrow$ Any vector $\boldsymbol{w}$ in V can be written as $\boldsymbol{w} = c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n$ in a unique way.

  The latter is the practical reason why we care so much about bases!

  V could be some abstract vector space (of polynomials or Fourier series), meaning that vectors are abstract objects and not just our usual column vectors. However, as soon as we pick a basis of V, then we can represent every (abstract) vector $\boldsymbol{w}$ by the (usual) column vector $(c_1, c_2, \ldots, c_n)^T$.

  This means all of our results can be used, too, when working with these abstract spaces!

Orthogonal bases

Theorem 57. Suppose that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are nonzero and pairwise orthogonal. Then $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent.

Proof. Suppose that

$$c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n = \boldsymbol{0}.$$

Take the dot product of $\boldsymbol{v}_1$ with both sides:

$$\begin{aligned} 0 &= \boldsymbol{v}_1 \cdot (c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n) \\ &= c_1\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 + c_2\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + \ldots + c_n\boldsymbol{v}_1 \cdot \boldsymbol{v}_n \\ &= c_1\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 = c_1\|\boldsymbol{v}_1\|^2 \end{aligned}$$

But $\|\boldsymbol{v}_1\| \neq 0$ and hence $c_1 = 0$.

Likewise, we find $c_2 = 0$, ..., $c_n = 0$. Hence, the vectors are independent. □

Comment. Note that this result is intuitively obvious: if the vectors were linearly dependent, then one of them could be written as a linear combination of the others. However, all these other vectors (and hence any combination of them) are orthogonal to it.

Definition 58. A basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ of a vector space V is an **orthogonal basis** if the vectors are (pairwise) orthogonal.

If, in addition, the basis vectors have length 1, then this is called an **orthonormal basis**.

Example 59. The standard basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an orthonormal basis for $\mathbb{R}^3$.

Example 60. Are the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ an orthogonal basis for $\mathbb{R}^3$? Is it orthonormal?

Solution. $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$.

So, this is an orthogonal basis.

Note that we do not need to check that the three vectors are independent. That follows from their orthogonality (see Theorem 57).

On the other hand, the vectors do not all have length 1, so that this basis is not orthonormal.

Normalize the vectors to produce an orthonormal basis.

Solution.

$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ has length $\sqrt{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} = \sqrt{2}$ $\Longrightarrow$ normalized: $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ has length $\sqrt{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} = \sqrt{2}$ $\Longrightarrow$ normalized: $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has length $\sqrt{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} = 1$ $\Longrightarrow$ is already normalized: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The resulting orthonormal basis is $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

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