

## The fundamental theorem

**Example 23.** The four **fundamental subspaces** associated with a matrix  $A$  are

$$\text{col}(A), \quad \text{row}(A), \quad \text{null}(A), \quad \text{null}(A^T).$$

Note that  $\text{row}(A) = \text{col}(A^T)$ . (In particular, we usually write vectors in  $\text{row}(A)$  as column vectors.)

**Definition 24.**  $\text{null}(A^T)$  is the **left null space** of  $A$ .

**Why that name?** Recall that, by definition  $\mathbf{x}$  is in  $\text{null}(A) \iff A\mathbf{x} = \mathbf{0}$ .

Likewise,  $\mathbf{x}$  is in  $\text{null}(A^T) \iff A^T\mathbf{x} = \mathbf{0} \iff \mathbf{x}^T A = \mathbf{0}$ .

[Recall that  $(AB)^T = B^T A^T$ . In particular,  $(A^T \mathbf{x})^T = \mathbf{x}^T A$ , which is what we used in the last equivalence.]

**Theorem 25. (Fundamental Theorem of Linear Algebra, Part I)**

Let  $A$  be an  $m \times n$  matrix of **rank**  $r$ .

- $\dim \text{col}(A) = r$  (subspace of  $\mathbb{R}^m$ )
- $\dim \text{row}(A) = r$  (subspace of  $\mathbb{R}^n$ )  $\text{row}(A) = \text{col}(A^T)$
- $\dim \text{null}(A) = n - r$  (subspace of  $\mathbb{R}^n$ )
- $\dim \text{null}(A^T) = m - r$  (subspace of  $\mathbb{R}^m$ )

**Example 26.** Let  $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$ . Determine bases for all four fundamental subspaces.

**Solution.** Make sure that, for such a simple matrix, you can see all of these that at a glance!

$$\text{col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}, \quad \text{row}(A) = \text{span} \left\{ \begin{bmatrix} 1 & 2 \end{bmatrix} \right\}, \quad \text{null}(A) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}, \quad \text{null}(A^T) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

**Example 27. (important observation)** For  $A$  as in the previous example, what do you notice about the basis vectors for  $\text{row}(A)$  and  $\text{null}(A)$ ? What about  $\text{col}(A)$  and  $\text{null}(A^T)$ ?

**Solution.** The basis vectors for  $\text{row}(A)$  and  $\text{null}(A)$  are orthogonal!  $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$

The same is true for the basis vectors for  $\text{col}(A)$  and  $\text{null}(A^T)$ :  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = 0$  and  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = 0$

Vectors in  $\text{null}(A)$  are orthogonal to vectors in  $\text{row}(A)$ .

For short,  $\text{null}(A)$  is orthogonal to  $\text{row}(A)$ .

**Why?** Suppose that  $\mathbf{x}$  is in  $\text{null}(A)$ . That is,  $A\mathbf{x} = \mathbf{0}$ .

But think about what  $A\mathbf{x} = \mathbf{0}$  means (row-product rule).

It means that the inner product of every row with  $\mathbf{x}$  is zero.

But that implies that  $\mathbf{x}$  is orthogonal to the row space.

**Definition 28.** As done in the observation above, we say that two subspaces  $V$  and  $W$  of  $\mathbb{R}^n$  are **orthogonal** if and only if every vector in  $V$  is orthogonal to every vector in  $W$ .

The **orthogonal complement** of  $W$  is the space  $W^\perp$  of all vectors that are orthogonal to  $W$ .

**Exercise.** Show that the orthogonal complement is indeed a vector space.

**Theorem 29. (Fundamental Theorem of Linear Algebra, Part II)**

- $\text{null}(A)$  is orthogonal to  $\text{row}(A)$ . (both subspaces of  $\mathbb{R}^n$ )

Note that  $\dim \text{null}(A) + \dim \text{row}(A) = n$ .

Hence, the two spaces are orthogonal complements.

- $\text{null}(A^T)$  is orthogonal to  $\text{col}(A)$ .

Again, the two spaces are orthogonal complements.

**Note.** The second part is just the first part with  $A$  replaced by  $A^T$ .

**Example 30.** Let  $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$ . Check that  $\text{null}(A)$  and  $\text{row}(A)$  are orthogonal complements.

**Solution.**  $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix} \xrightarrow[\begin{smallmatrix} R_3 - 3R_1 \Rightarrow R_3 \\ \rightsquigarrow \end{smallmatrix}]{\begin{smallmatrix} R_2 - 2R_1 \Rightarrow R_2 \\ R_3 - 3R_1 \Rightarrow R_3 \end{smallmatrix}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow[\rightsquigarrow]{R_3 - \frac{3}{2}R_2 \Rightarrow R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\rightsquigarrow]{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Hence,  $\text{null}(A) = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}$ ,  $\text{row}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

The basis vectors are orthogonal because

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 0, \quad \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0.$$

**Note.** Because  $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$  is orthogonal to both basis vectors, it is orthogonal to every vector in the row space.

Vectors in  $\text{row}(A)$  are of the form  $\mathbf{v} = a \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ . Then,  $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \mathbf{v} = a \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$ .

**Conclusion.** Hence,  $\text{null}(A)$  and  $\text{row}(A)$  are indeed orthogonal spaces.

In fact,  $\text{null}(A)$  and  $\text{row}(A)$  are orthogonal complements.

That is because  $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  are orthogonal, hence independent, and thus a basis of all of  $\mathbb{R}^3$ .

**Example 31. (extra)** Determine bases for all four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 1 \\ 3 & 6 & 0 & 1 \end{bmatrix}.$$

Verify all parts of the Fundamental Theorem, especially that  $\text{null}(A)$  and  $\text{row}(A)$  (as well as  $\text{null}(A^T)$  and  $\text{col}(A)$ ) are orthogonal complements.

**Partial solution.** One can almost see that  $\text{rank}(A) = 3$ . Hence, the dimensions of the fundamental subspaces are ...

Any serious linear algebra problems are done by a machine. Let us see how to use the open-source computer algebra system **Sage** to do basic computations for us.

Sage is freely available at [sagemath.org](http://sagemath.org). Instead of installing it locally (it's huge!) we can conveniently use it in the cloud at [cocalc.com](http://cocalc.com) from any browser. For short computations, like the one below, you can also just use the input field on our course website.

Sage is built as a **Python** library, so any Python code is valid. Here, we will just use it as a fancy calculator.

Let's revisit Example 30 and let Sage do the work for us:

```
Sage] A = matrix([[1,2,1],[2,4,0],[3,6,0]])
```

```
Sage] A.rref()
```

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Similarly, if we wanted to compute a basis for  $\text{null}(A^T)$ , we can simply do:

```
Sage] A.transpose().rref()
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Here's some other standard things we might be interested in (compare with Example 10):

```
Sage] A = matrix([[4,0,2],[2,2,2],[1,0,3]])
```

```
Sage] A.eigenvalues()
```

```
[5, 2, 2]
```

```
Sage] A.eigenvectors_right()
```

$$\left[ \left( 5, \left[ \left( 1, 1, \frac{1}{2} \right) \right], 1 \right), \left( 2, \left[ (1, 0, -1), (0, 1, 0) \right], 2 \right) \right]$$

```
Sage] A.eigenmatrix_right()
```

$$\left( \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ \frac{1}{2} & -1 & 0 \end{bmatrix} \right)$$

```
Sage] A.rank()
```

```
3
```

```
Sage] A.determinant()
```

```
20
```

```
Sage] A.inverse()
```

$$\begin{bmatrix} \frac{3}{10} & 0 & -\frac{1}{5} \\ -\frac{1}{5} & \frac{1}{2} & -\frac{1}{5} \\ -\frac{1}{10} & 0 & \frac{2}{5} \end{bmatrix}$$