Orthogonality

The inner product and distances

Definition 15. The **inner product** (or **dot product**) of $\boldsymbol{v}$, $\boldsymbol{w}$ in $\mathbb{R}^n$:

$$\boldsymbol{v} \cdot \boldsymbol{w} \;=\; \boldsymbol{v}^T \boldsymbol{w} = v_1 w_1 + \ldots + v_n w_n.$$

Because we can think of this as a special case of the matrix product, it satisfies the basic rules like associativity and distributivity.

In addition: $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$.

Example 16. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = 2 - 2 + 12 = 12$

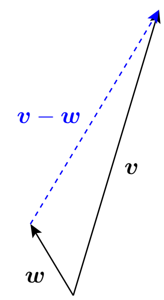
Definition 17.

- The **norm** (or **length**) of a vector $\boldsymbol{v}$ in $\mathbb{R}^n$ is

$$\|\boldsymbol{v}\| \;=\; \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + \ldots + v_n^2}.$$

- The **distance** between points $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ is

$$\operatorname{dist}(\boldsymbol{v}, \boldsymbol{w}) \;=\; \|\boldsymbol{v} - \boldsymbol{w}\|.$$



Example 18. For instance, in $\mathbb{R}^2$, $\operatorname{dist}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \left\| \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix} \right\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Example 19. Write $\|\boldsymbol{v} - \boldsymbol{w}\|^2$ as a dot product, and multiply it out.

Solution. $\|\boldsymbol{v} - \boldsymbol{w}\|^2 = (\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|^2 - 2\boldsymbol{v} \cdot \boldsymbol{w} + \|\boldsymbol{w}\|^2$

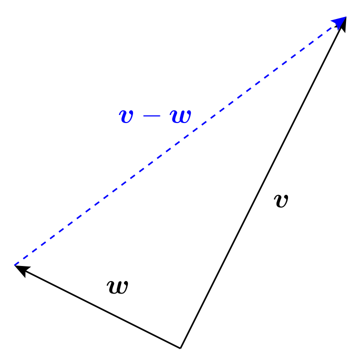
Comment. This is a vector version of $(x - y)^2 = x^2 - 2xy + y^2$.

The reason we were careful and first wrote $-\boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v}$ before simplifying it to $-2\boldsymbol{v} \cdot \boldsymbol{w}$ is that we should not take rules such as $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$ for granted. For instance, for the cross product $\boldsymbol{v} \times \boldsymbol{w}$, that you may have seen in Calculus, we have $\boldsymbol{v} \times \boldsymbol{w} \neq \boldsymbol{w} \times \boldsymbol{v}$ (instead, $\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$).

Orthogonal vectors

Definition 20. $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ are **orthogonal** if

$$\boldsymbol{v} \cdot \boldsymbol{w} = 0.$$



Why? How is this related to our understanding of right angles?

Pythagoras!

$\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal

$\iff \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = \underbrace{\|\boldsymbol{v} - \boldsymbol{w}\|^2}_{\substack{= \|\boldsymbol{v}\|^2 - 2\boldsymbol{v} \cdot \boldsymbol{w} + \|\boldsymbol{w}\|^2 \\ \text{(by previous example)}}}$

$\iff -2\boldsymbol{v} \cdot \boldsymbol{w} = 0$

$\iff \boldsymbol{v} \cdot \boldsymbol{w} = 0$

Example 21. Determine a basis for the **orthogonal complement** of (the span of) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

What are we looking for? The orthogonal complement of $\boldsymbol{v}$ consists of all vectors that are orthogonal to $\boldsymbol{v}$. More generally, the orthogonal complement of a space V consists of all vectors that are orthogonal to every vector in V.

Solution. (staring/intution) We are working in 3-dimensional space and already have 1 vector. The vectors orthogonal to it lie in a $3-1=2$-dimensional space (a plane).

Two of the vectors orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ are $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Knowing that the orthogonal complement has dimension 2, we conclude that $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is a basis.

In other words, the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$ is $\operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}$.

[Note how the dimensions add up to the dimension of the entire space: $1+2=3$.]

Solution. (professional) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$ (dot product!) is the same as $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$ (matrix product!).

Hence, the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$ is the same as $\operatorname{null}(\begin{bmatrix} 1 & 2 & 1 \end{bmatrix})$.

Computing a basis for $\operatorname{null}(\begin{bmatrix} 1 & 2 & 1 \end{bmatrix})$ is easy since $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$ is already in RREF.

Note that the general solution to $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\boldsymbol{x} = 0$ is $\begin{bmatrix} -2s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

A basis for $\operatorname{null}(\begin{bmatrix} 1 & 2 & 1 \end{bmatrix})$ therefore is $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. (Check that these are indeed orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$!)

Example 22. Determine a basis for the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}\right\}$.

Solution. We are looking for vectors $\boldsymbol{x}$ such that $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$ and $\begin{bmatrix} 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$.

The two equations can be combined into a single one: $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$.

In other words, the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}\right\}$ is the same as $\operatorname{null}\left(\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}\right)$.

It remains to compute a basis for that null space:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \overset{R_2 - 3R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \end{bmatrix} \quad \overset{\text{back-substitution}}{\rightsquigarrow} \quad \begin{bmatrix} -3/5 s \\ -1/5 s \\ s \end{bmatrix}$$

Hence, a basis for the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}\right\}$ is $\begin{bmatrix} -3/5 \\ -1/5 \\ 1 \end{bmatrix}$.

Check. $\begin{bmatrix} -3/5 \\ -1/5 \\ 1 \end{bmatrix}$ is indeed orthogonal to both $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

Just to make sure. Why was it clear that the orthogonal complement is 1-dimensional?

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