

Example 7.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 2 & 3 \\ -16 & 5 & 6 \\ -25 & 8 & 9 \end{bmatrix}$$

Multiplication (on the right) with that “almost identity matrix” is performing the column operation $C_1 - 4C_2 \Rightarrow C_1$ (i.e. -4 times the second column is added to the first column).

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$$

Multiplication (on the left) with the same matrix is performing the row operation $R_2 - 4R_1 \Rightarrow R_2$.

Comment (again). The row operations we are doing during Gaussian elimination can all be realized by multiplying (on the left) with “almost identity matrices”.

These matrices are called **elementary matrices** (they are obtained by performing a single elementary row operation on an identity matrix).

Elementary matrices are **invertible** because elementary row operations are reversible:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & \\ & \frac{1}{2} & \\ & & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 8. Determine the LU decomposition of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 - 3R_1 \Rightarrow R_2} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ translates into $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$.

Since $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ (no calculation needed!), we therefore have

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}.$$

Review: Eigenvalues and eigenvectors

If $Ax = \lambda x$ (and $x \neq 0$), then x is an **eigenvector** of A with **eigenvalue** λ (just a number).

Note that for the equation $Ax = \lambda x$ to make sense, A needs to be a square matrix (i.e. $n \times n$).

Key observation:

$$\begin{aligned} Ax &= \lambda x \\ \iff Ax - \lambda x &= 0 \\ \iff (A - \lambda I)x &= 0 \end{aligned}$$

This homogeneous system has a nontrivial solution x if and only if $\det(A - \lambda I) = 0$.

To find eigenvectors and eigenvalues of A :

(a) First, find the eigenvalues λ by solving $\det(A - \lambda I) = 0$.

$\det(A - \lambda I)$ is a polynomial in λ , called the **characteristic polynomial** of A .

(b) Then, for each eigenvalue λ , find corresponding eigenvectors by solving $(A - \lambda I)x = 0$.

More precisely, we find a basis of eigenvectors for the λ -**eigenspace** $\text{null}(A - \lambda I)$.

Example 9. $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ has one eigenvector that is “easy” to see. Do you see it?

Solution. Note that $A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Hence, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a 2-eigenvector.

Just for contrast. Note that $A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \neq \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Hence, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not an eigenvector.

Suppose that A is $n \times n$ and has independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$.
Then A can be **diagonalized** as $A = PDP^{-1}$, where

- the columns of P are the eigenvectors, and
- the diagonal matrix D has the eigenvalues on the diagonal

Example 10. Let $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$.

- (a) Find the eigenvalues and bases for the eigenspaces of A .
- (b) Diagonalize A . That is, determine matrices P and D such that $A = PDP^{-1}$.

Solution.

- (a) By expanding by the second column, we find that the characteristic polynomial $\det(A - \lambda I)$ is

$$\begin{vmatrix} 4-\lambda & 0 & 2 \\ 2 & 2-\lambda & 2 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)[(4-\lambda)(3-\lambda) - 2] = (2-\lambda)^2(5-\lambda).$$

Hence, the eigenvalues are $\lambda = 2$ (with multiplicity 2) and $\lambda = 5$.

For $\lambda = 5$, the eigenspace $\text{null}\left(\begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 1 & 0 & -2 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. (Fill in the details if necessary!)

For $\lambda = 2$, the eigenspace $\text{null}\left(\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

[The row-reduced echelon form (RREF) of $\begin{bmatrix} 2 & 0 & 2 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.]

- (b) A possible choice is $P = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.