Preparing for the Final

MATH 316 — Linear Algebra II
Monday, April 30

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. The final exam will be comprehensive, that is, it will cover the material of the whole semester.

(a) Redo the practice problems for both midterms.

(b) Retake both midterms. (For your convenience, these are posted with and without solutions.)

(c) Redo the online homeworks #7 and #8.

(d) Do the problems below. (Solutions will be posted soon.)

Problem 2. Solve the initial value problem

$$\boldsymbol{y}' = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Solution. The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

- Diagonalize $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$:
  - $\begin{vmatrix} -\lambda & -2 \\ -4 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 8$, so the eigenvalues are $-2, 4$
  - $\lambda = 4$ has eigenspace $\operatorname{null}\left(\begin{bmatrix} -4 & -2 \\ -4 & -2 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$
  - $\lambda = -2$ has eigenspace $\operatorname{null}\left(\begin{bmatrix} 2 & -2 \\ -4 & 4 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$
  - Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$.
- Compute the solution $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned}
\boldsymbol{y} &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\
&= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3e^{-2t} \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix}
\end{aligned}$$

□

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Problem 3.

(a) Convert the third-order differential equation

$$y''' = 6y'' - 3y' - 10y, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3$$

to a system of first-order differential equations.

(b) Solve the original differential equation by solving the system.

Solution.

(a) Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' = 6y'' - 3y' - 10y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -10y_1 - 3y_2 + 6y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -3 & 6 \end{bmatrix}\boldsymbol{y}, \quad \boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(b) Recall that the solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$.

- First, to compute e^{At} for $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -3 & 6 \end{bmatrix}$, we need to diagonalize A.
  - The eigenvalues of A are $\lambda = 5, 2, -1$.
  - The 5-eigenspace $\operatorname{null}\left(\begin{bmatrix} -5 & 1 & 0 \\ 0 & -5 & 1 \\ -10 & -3 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 5 \\ 25 \end{bmatrix}$.
  - The 2-eigenspace $\operatorname{null}\left(\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ -10 & -3 & 4 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$.
  - The -1-eigenspace $\operatorname{null}\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -10 & -3 & 7 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

  Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & & \\ & 2 & \\ & & -1 \end{bmatrix}$.

- Then, we compute the solution $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned}
\boldsymbol{y} = e^{At}\boldsymbol{y}_0 &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix}\begin{bmatrix} e^{5t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix}\begin{bmatrix} e^{5t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}\frac{1}{18}\begin{bmatrix} -1 \\ 20 \\ -1 \end{bmatrix} \\
&= \frac{1}{18}\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix}\begin{bmatrix} -e^{5t} \\ 20e^{2t} \\ -e^{-t} \end{bmatrix} \\
&= \frac{1}{18}\begin{bmatrix} -e^{5t} + 20e^{2t} - e^{-t} \\ -5e^{5t} + 40e^{2t} + e^{-t} \\ -25e^{5t} + 80e^{2t} - e^{-t} \end{bmatrix}
\end{aligned}$$

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In particular, the original differential equation is solved by $y(t) = \frac{1}{18}(-e^{5t} + 20e^{2t} - e^{-t})$.

Comment. To compute $\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} -1 \\ 20 \\ -1 \end{bmatrix}$, we solve $\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ to find $\boldsymbol{x} = \frac{1}{18} \begin{bmatrix} -1 \\ 20 \\ -1 \end{bmatrix}$.

Next comment. Obviously, computations will be more pleasant on the exam.

□

Problem 4.

(a) What are the possible Jordan normal forms of a 6×6 matrix with eigenvalues $7, 7, 3, 3, 3, 3$?

(b) How many different Jordan normal forms are there for a 10×10 matrix with eigenvalues $8, 6, 6, 2, 2, 2, 1, 1, 1, 1$?

Solution.

(a) There are $2 \cdot 5 = 10$ possibilities:

$$\begin{bmatrix} 7 &&&&& \\ & 7 &&&& \\ && 3 &&& \\ &&& 3 && \\ &&&& 3 & \\ &&&&& 3 \end{bmatrix}, \begin{bmatrix} 7 &&&&& \\ & 7 &&&& \\ && 3 & 1 && \\ &&& 3 && \\ &&&& 3 & \\ &&&&& 3 \end{bmatrix}, \begin{bmatrix} 7 &&&&& \\ & 7 &&&& \\ && 3 & 1 && \\ &&& 3 && \\ &&&& 3 & 1 \\ &&&&& 3 \end{bmatrix}, \begin{bmatrix} 7 &&&&& \\ & 7 &&&& \\ && 3 & 1 && \\ &&& 3 & 1 & \\ &&&& 3 & \\ &&&&& 3 \end{bmatrix}, \begin{bmatrix} 7 &&&&& \\ & 7 &&&& \\ && 3 & 1 && \\ &&& 3 & 1 & \\ &&&& 3 & 1 \\ &&&&& 3 \end{bmatrix},$$

$$\begin{bmatrix} 7 & 1 &&&& \\ & 7 &&&& \\ && 3 &&& \\ &&& 3 && \\ &&&& 3 & \\ &&&&& 3 \end{bmatrix}, \begin{bmatrix} 7 & 1 &&&& \\ & 7 &&&& \\ && 3 & 1 && \\ &&& 3 && \\ &&&& 3 & \\ &&&&& 3 \end{bmatrix}, \begin{bmatrix} 7 & 1 &&&& \\ & 7 &&&& \\ && 3 & 1 && \\ &&& 3 && \\ &&&& 3 & 1 \\ &&&&& 3 \end{bmatrix}, \begin{bmatrix} 7 & 1 &&&& \\ & 7 &&&& \\ && 3 & 1 && \\ &&& 3 & 1 & \\ &&&& 3 & \\ &&&&& 3 \end{bmatrix}, \begin{bmatrix} 7 & 1 &&&& \\ & 7 &&&& \\ && 3 & 1 && \\ &&& 3 & 1 & \\ &&&& 3 & 1 \\ &&&&& 3 \end{bmatrix}$$

(b) There are $1 \cdot 2 \cdot 3 \cdot 5 = 30$ possible different Jordan normal forms. □

Problem 5. Find the best approximation of $f(x) = x$ on the interval $[0, 4]$ using a function of the form $y = a + b\sqrt{x}$.

Solution. The best approximation we are looking for is the orthogonal projection of $f(x)$ onto $\operatorname{span}\{1, \sqrt{x}\}$, where the dot product of functions is

$$\langle f, g \rangle = \int_0^4 f(t) g(t) \mathrm{d}t.$$

To find an orthogonal basis for $\operatorname{span}\{1, \sqrt{x}\}$, following Gram–Schmidt, we compute

$$\sqrt{x} - \begin{pmatrix} \text{projection of} \\ \sqrt{x} \text{ onto } 1 \end{pmatrix} = \sqrt{x} - \frac{\langle \sqrt{x}, 1 \rangle}{\langle 1, 1 \rangle} 1 = \sqrt{x} - \frac{4}{3}.$$

In the last step, we used that

$$\langle 1, 1 \rangle = \int_0^4 1 \mathrm{d}t = 4, \quad \langle \sqrt{x}, 1 \rangle = \int_0^4 \sqrt{t} \mathrm{d}t = \left[\frac{1}{3/2} t^{3/2} \right]_0^4 = \frac{16}{3}.$$

Hence, $1, \sqrt{x} - \frac{4}{3}$ is an orthogonal basis for $\operatorname{span}\{1, \sqrt{x}\}$.

The orthogonal projection of $f: [0, 4] \to \mathbb{R}$ onto $\operatorname{span}\{1, \sqrt{x}\} = \operatorname{span}\left\{1, \sqrt{x} - \frac{4}{3}\right\}$ therefore is

$$\frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} 1 + \frac{\left\langle f, \sqrt{x} - \frac{4}{3} \right\rangle}{\left\langle \sqrt{x} - \frac{4}{3}, \sqrt{x} - \frac{4}{3} \right\rangle} \left(\sqrt{x} - \frac{4}{3} \right) = \frac{1}{4} \int_0^4 f(t) \mathrm{d}t + \frac{9}{8} \left(\sqrt{x} - \frac{4}{3} \right) \int_0^4 f(t) \left(\sqrt{t} - \frac{4}{3} \right) \mathrm{d}t.$$

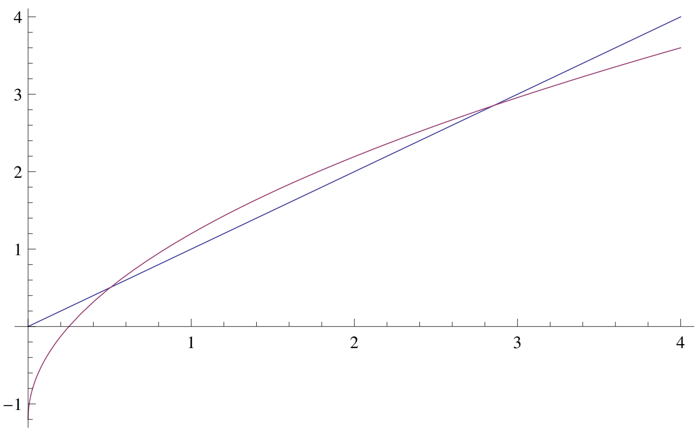
Here, we used that

$$\left\langle \sqrt{x}-\frac{4}{3},\sqrt{x}-\frac{4}{3}\right\rangle = \int_0^4 \left(\sqrt{t}-\frac{4}{3}\right)^2 \mathrm{d}t = \int_0^4 \left(t-\frac{8}{3}\sqrt{t}+\frac{16}{9}\right)\mathrm{d}t = \left[\frac{t^2}{2}-\frac{16}{9}t^{3/2}+\frac{16}{9}t\right]_0^4 = 8-\frac{128}{9}+\frac{64}{9} = \frac{8}{9}.$$

In our case, this best approximation is

$$\begin{aligned} & \frac{1}{4}\int_0^4 t\mathrm{d}t + \frac{9}{8}\left(\sqrt{x}-\frac{4}{3}\right)\int_0^4 t\left(\sqrt{t}-\frac{4}{3}\right)\mathrm{d}t \\ = \ & \frac{1}{4}\left[\frac{t^2}{2}\right]_0^4 + \frac{9}{8}\left(\sqrt{x}-\frac{4}{3}\right)\left[\frac{2}{5}t^{5/2}-\frac{2}{3}t^2\right]_0^4 = 2+\frac{12}{5}\left(\sqrt{x}-\frac{4}{3}\right) = \frac{12}{5}\sqrt{x}-\frac{6}{5}. \end{aligned}$$

The plot below confirms the quality of this linear approximation:



Comment. Computations will be more pleasant on the exam. Do make sure though that the ideas are clear. □

Problem 6. Give a basis for the space of all polynomials $p(x)$ of degree 4 or less such that $p(0)=p(1)$ and $p'(-1)=0$.

Solution. Let us start with the basis $1, x, x^2, x^3, x^4$ for the space of all polynomials $p(x)$ of degree 4 or less.

Then, we can identify the polynomial $p(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$ with the vector $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$.

The condition $p(0)=p(1)$ translates into $a_0=a_0+a_1+a_2+a_3+a_4$, that is, $a_1+a_2+a_3+a_4=0$.
Since $p'(x)=a_1+2a_2x+3a_3x^2+4a_4x^3$, the condition $p'(-1)=0$ translates into $a_1-2a_2+3a_3-4a_4=0$.
In other words, the space of all polynomials $p(x)$ of degree 4 or less such that $p(0)=p(1)$ and $p'(-1)=0$ translates into $\mathrm{null}(M)$ with $M=\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 3 & -4 \end{bmatrix}$.
Since

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 3 & -4 \end{bmatrix} \overset{R_2-R_1\Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -3 & 2 & -5 \end{bmatrix} \overset{-\frac{1}{3}R_2\Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{5}{3} \end{bmatrix} \overset{R_1-R_2\Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 0 & 1 & 0 & \frac{5}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{5}{3} \end{bmatrix},$$

the general solution to $M\boldsymbol{x}=\mathbf{0}$ is $\begin{bmatrix} s_1 \\ -\frac{5}{3}s_2+\frac{2}{3}s_3 \\ \frac{2}{3}s_2-\frac{5}{3}s_3 \\ s_2 \\ s_3 \end{bmatrix}$. In particular, a basis for $\mathrm{null}(M)$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5/3 \\ 2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2/3 \\ -5/3 \\ 0 \\ 1 \end{bmatrix}$.

The corresponding polynomials are 1, $x^3+\frac{2}{3}x^2-\frac{5}{3}x$ and $x^4-\frac{5}{3}x^2+\frac{2}{3}x$.

Check. Check that these polynomials indeed satisfy $p(0)=p(1)$ and $p'(-1)=0$.

Comment. Let's note that it was to be expected from the beginning that the space is 3-dimensional. The space of all polynomials $p(x)$ of degree 4 or less has dimension 5. Since we impose 2 (independent) conditions, the dimension of our space is $5-2=3$.

□

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