Quiz #3

MATH 316 — Linear Algebra II
Wednesday, Mar 8

Please print your name:

Problem 1. Determine the SVD of $A = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}$.

Solution. $A^T A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$ has characteristic polynomial $(5-\lambda)(8-\lambda) - 4 = \lambda^2 - 13\lambda + 36 = (\lambda - 9)(\lambda - 4)$.

The 9-eigenspace, that is $\operatorname{null}\left(\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}\right)$, is spanned by $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

The 4-eigenspace, that is $\operatorname{null}\left(\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\right)$, is spanned by $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Hence, we conclude that $V = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 3 & \\ & 2 \end{bmatrix}$.

From $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$, we find $\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix} \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Likewise, $\boldsymbol{u}_2 = \frac{1}{\sigma_2} A\boldsymbol{v}_2 = \frac{1}{2}\begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix} \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Hence, $U = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

In conclusion, $A = U\Sigma V^T$ with $U = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 3 & \\ & 2 \end{bmatrix}$, $V = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$. □

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