Quiz #2

MATH 316 — Linear Algebra II
Wednesday, Feb 1

Please print your name:

Problem 1.

(a) Find the least squares solution to $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$.

(b) What is the orthogonal projection of $\begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$ onto span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$?

(c) Determine the least squares line for the data points $(1,1), (0,6), (2,2)$.

Solution. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$.

(a) Since $A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$ and $A^T \boldsymbol{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$, the normal equations $A^T A \hat{\boldsymbol{x}} = A^T \boldsymbol{b}$ are

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \hat{\boldsymbol{x}} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}.$$

Solving, we immediately find $\hat{\boldsymbol{x}} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$.

For instance. When working by hand, a convenient way is $\hat{\boldsymbol{x}} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 5 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$.

Check. The error $A\hat{\boldsymbol{x}} - \boldsymbol{b} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ is orthogonal to col(A): $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 0$

(b) The orthogonal projection of $\begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$ onto span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$ is $A\hat{\boldsymbol{x}} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$.

(c) We need to determine the values a, b for the least squares line $y = a + bx$. The equations $a + bx_i = y_i$ translate into the system

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad \text{that is,} \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}.$$

We have already computed that the least squares solution to that system is $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$.

Hence, the least squares line is $y = 5 - 2x$.

□

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