

Quiz #1

Please print your name:

Problem 1. Diagonalize $A = \begin{bmatrix} 5 & -2 & -2 \\ 4 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$. (That is, determine matrices P and D such that $A = PDP^{-1}$.)

Solution. By expanding by the third row, we find that the characteristic polynomial is

$$\begin{vmatrix} 5-\lambda & -2 & -2 \\ 4 & -1-\lambda & -2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 5-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = (1-\lambda)^2(3-\lambda).$$

Hence, the eigenvalues are 1, 1, 3.

- For $\lambda = 3$, the eigenspace $\text{null}\left(\begin{bmatrix} 2 & -2 & -2 \\ 4 & -4 & -2 \\ 0 & 0 & -2 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.
- For $\lambda = 1$, the eigenspace $\text{null}\left(\begin{bmatrix} 4 & -2 & -2 \\ 4 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Thus, if we choose $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & & \\ & 1 & \\ & & 1 \end{bmatrix}$, then $A = PDP^{-1}$. (There is, of course, other possible choices.)

Check that we got it right. We can check this by verifying $AP = PD$:

$$\begin{bmatrix} 5 & -2 & -2 \\ 4 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

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