Preparing for Midterm #2

MATH 316 — Linear Algebra II
Wednesday, Mar 29

Please print your name:

Problem 1.

(a) Do the practice problems that were compiled from the examples from our lectures.

In particular, fill in all the conceptual empty boxes.

To save time, you don't need to work through all details. However, make sure that you know how to do each problem.

(b) Retake Quiz 3.

(c) Do the problems below. (Solutions will be posted soon.)

Bonus challenge. Let me know about any typos you spot in our lecture sketches or the posted solutions (surely, there should be some). Any typo that is not yet fixed on our course website by the time you send it to me, is worth a small bonus.

Problem 2. Consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$.

(a) Determine the SVD of A.

(b) Determine the pseudoinverse of A.

(c) Find the smallest solution to $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(Then, as a mild check, compare its norm to the obvious solution $\boldsymbol{x} = [\, 1 \;\; 1 \;\; 0 \,]^T$.)

(d) What is the best approximation to A using a 2×3 matrix with rank 1?

Solution.

(a) $A^T A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ has characteristic polynomial

$$\begin{aligned} \det\left(\begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix}\right) &= 0 - 1 \cdot \det\left(\begin{bmatrix} 2-\lambda & 0 \\ 1 & 1 \end{bmatrix}\right) + (2-\lambda)\det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\right) \\ &= -(2-\lambda) + (2-\lambda)\underbrace{((2-\lambda)(1-\lambda) - 1)}_{=\lambda^2 - 3\lambda + 1} \\ &= (2-\lambda)(\lambda^2 - 3\lambda) = (2-\lambda)\lambda(\lambda - 3). \end{aligned}$$

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Hence, the eigenvalues are $0, 2, 3$.

- The 0-eigenspace $\text{null}\left(\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

- The 2-eigenspace $\text{null}\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

- The 3-eigenspace $\text{null}\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Therefore, $V = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$.

Next, $\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{u}_2 = \frac{1}{\sigma_2} A\boldsymbol{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Hence, $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

In summary, $A = U\Sigma V^T$ with $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$, $V = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$.

(b) The pseudoinverse of A is

$$A^+ = V\Sigma^+ U^T = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/2 \\ 1/3 & 0 \\ 1/3 & -1/2 \end{bmatrix}.$$

(c) The smallest solution to $A\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is

$$\boldsymbol{x} = A^+\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/2 \\ 1/3 & 0 \\ 1/3 & -1/2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 2/3 \\ 1/6 \end{bmatrix}.$$

(For comparison, $\|\boldsymbol{x}\| = \sqrt{11/6} \approx 1.354$ is indeed less than $\|[\,1\ \ 1\ \ 0\,]^T\| = \sqrt{2} \approx 1.414$.)

(d) The best approximation to

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

using a 2×3 matrix with rank 1 (that is, we keep 1 singular value) is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

□

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Problem 3. Solve the initial value problem

$$\boldsymbol{y}' = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Solution. The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

- Diagonalize $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$:
  - $\begin{vmatrix} -\lambda & -2 \\ -4 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 8$, so the eigenvalues are $-2, 4$
  - $\lambda = 4$ has eigenspace $\operatorname{null}\left(\begin{bmatrix} -4 & -2 \\ -4 & -2 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$
  - $\lambda = -2$ has eigenspace $\operatorname{null}\left(\begin{bmatrix} 2 & -2 \\ -4 & 4 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$
  - Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$.
- Compute the solution $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y} &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3e^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix} \end{aligned}$$

□

Problem 4.

(a) Convert the third-order differential equation

$$y''' = 6y'' - 3y' - 10y, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3$$

to a system of first-order differential equations.

(b) Solve the original differential equation by solving the system.

Solution.

(a) Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

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Then, $y''' = 6y'' - 3y' - 10y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -10y_1 - 3y_2 + 6y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -3 & 6 \end{bmatrix} \boldsymbol{y}$, $\quad \boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(b) Recall that the solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$.

- First, to compute e^{At} for $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -3 & 6 \end{bmatrix}$, we need to diagonalize A.

  - The eigenvalues of A are $\lambda = 5, 2, -1$.

  - The 5-eigenspace null$\left(\begin{bmatrix} -5 & 1 & 0 \\ 0 & -5 & 1 \\ -10 & -3 & 1 \end{bmatrix} \right)$ has basis $\begin{bmatrix} 1 \\ 5 \\ 25 \end{bmatrix}$.

  - The 2-eigenspace null$\left(\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ -10 & -3 & 4 \end{bmatrix} \right)$ has basis $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$.

  - The -1-eigenspace null$\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -10 & -3 & 7 \end{bmatrix} \right)$ has basis $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

  Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & & \\ & 2 & \\ & & -1 \end{bmatrix}$.

- Then, we compute the solution $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y} = e^{At}\boldsymbol{y}_0 &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \frac{1}{18} \begin{bmatrix} -1 \\ 20 \\ -1 \end{bmatrix} \\ &= \frac{1}{18} \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix} \begin{bmatrix} -e^{5t} \\ 20e^{2t} \\ -e^{-t} \end{bmatrix} \\ &= \frac{1}{18} \begin{bmatrix} -e^{5t} + 20e^{2t} - e^{-t} \\ -5e^{5t} + 40e^{2t} + e^{-t} \\ -25e^{5t} + 80e^{2t} - e^{-t} \end{bmatrix} \end{aligned}$$

  In particular, the original differential equation is solved by $y(t) = \frac{1}{18}(-e^{5t} + 20e^{2t} - e^{-t})$.

  Comment. To compute $\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} -1 \\ 20 \\ -1 \end{bmatrix}$, we solve $\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ to find $\boldsymbol{x} = \frac{1}{18} \begin{bmatrix} -1 \\ 20 \\ -1 \end{bmatrix}$.

  Next comment. Obviously, computations will be more pleasant on the exam.

□

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Problem 5. True or false? (As usual, "true" means that the statement is always true.) Explain!

(a) The product of two orthogonal matrices is orthogonal.

(b) $A^T A$ is symmetric for any matrix A.

(c) $A A^T$ is symmetric for any matrix A.

(d) A real $n \times n$ matrix A has real eigenvalues.

(e) The determinant of A is equal to the product of the singular values of A.

(f) The determinant of A is equal to the product of the eigenvalues of A.

(g) If the matrix A is symmetric, then $\langle A\boldsymbol{v}, \boldsymbol{w}\rangle = \langle \boldsymbol{v}, A\boldsymbol{w}\rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

(h) If the matrix A is orthogonal, then $\langle A\boldsymbol{v}, A\boldsymbol{w}\rangle = \langle \boldsymbol{v}, \boldsymbol{w}\rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

(i) If $\boldsymbol{v}$ and $\boldsymbol{w}$ are eigenvectors of A with different eigenvalues, then $\langle \boldsymbol{v}, \boldsymbol{w}\rangle = \boldsymbol{0}$.

(j) A is invertible if and only if the only solution to $A\boldsymbol{x} = \boldsymbol{0}$ is $\boldsymbol{x} = \boldsymbol{0}$.

(k) An $n \times n$ matrix A has eigenvalue 0 if and only if it has singular value 0.

(l) An $n \times n$ matrix A has eigenvalue 1 if and only if it has singular value 1.

(m) An $n \times n$ matrix A is singular if and only if 0 is an eigenvalue of A.

(n) An $n \times n$ matrix A is singular if and only if 0 is a singular value of A.

(o) Every symmetric real $n \times n$ matrix A is diagonalizable.

(p) Every symmetric real $n \times n$ matrix A is invertible.

Solution.

(a) True.

If $A^T = A^{-1}$ and $B^T = B^{-1}$, then $(AB)^T = B^T A^T = B^{-1} A^{-1} = (AB)^{-1}$.

(b) True.

(c) True.

(d) False, because this is not true for all matrices. (Take, for instance, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.)

However, by the spectral theorem, a symmetric real $n \times n$ matrix A must have real eigenvalues.

(e) False, but almost true.

Recall that the singular values are all nonnegative, whereas the determinant can be negative.

On the other hand, the absolute value of the determinant of A equals the absolute value of the product of the singular values of A. (Both U and V in $A = U\Sigma V^T$ have determinant ± 1 because they are orthogonal.)

(f) True.

(g) True.

Actually, a matrix A is symmetric if and only if $\langle A\boldsymbol{v}, \boldsymbol{w}\rangle = \langle \boldsymbol{v}, A\boldsymbol{w}\rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

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(h) True.

Actually, a matrix A is orthogonal if and only if $\langle A\boldsymbol{v}, A\boldsymbol{w}\rangle = \langle \boldsymbol{v}, \boldsymbol{w}\rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

(i) False, because this is not true for all matrices.

However, the statement is true for symmetric matrices by the spectral theorem.

(j) True.

(k) True. Both statements are equivalent to A not being invertible.

(l) False.

(m) True.

(n) True.

(o) True. (That's part of the spectral theorem.)

(p) False. □

Problem 6.

(a) If A has λ-eigenvalue $\boldsymbol{v}$, then A^3 has [ ].

(b) A is singular if and only if [$\dim \operatorname{null}(A)$].

(c) The eigenvalues of a 5×5 matrix for orthogonally projecting onto a 3-dimensional subspace are [ ].

(d) Suppose A is the 3×3 matrix of a reflection through a plane (containing the origin).

Then $\det(A) =$ [ ], and the eigenvalues of A are [ ].

(e) What exactly does it mean for a matrix A to have full column rank?

(f) Precisely state the spectral theorem.

Solution.

(a) If A has λ-eigenvalue $\boldsymbol{v}$, then A^3 has λ^3-eigenvalue $\boldsymbol{v}$.

(b) A is singular (i.e. not invertible) if and only if $\dim \operatorname{null}(A) > 0$.

(c) The eigenvalues of a 5×5 matrix for orthogonally projecting onto a 3-dimensional subspace are $1, 1, 1, 0, 0$.

(d) Suppose A is the 3×3 matrix of a reflection through a plane (containing the origin).

Then $\det(A) = -1$, and the eigenvalues of A are $1, 1, -1$.

(e) A matrix A has full column rank if its rank equals the number of columns.

(f) A symmetric (real) matrix A can always be diagonalized. Moreover, all eigenvalues are real and the eigenspaces are orthogonal. □