

# Preparing for Midterm #1

Please print your name:

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## Problem 1.

(a) Using Gram–Schmidt, obtain an orthonormal basis for  $W = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ .

(b) Determine the orthogonal projection of  $\begin{bmatrix} 2 \\ 6 \\ -1 \\ 3 \end{bmatrix}$  onto  $W$ .

(c) Determine the  $QR$  decomposition of the matrix  $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ .

(d) Determine a basis for the orthogonal complement  $W^\perp$ .

## Solution.

(a) Let  $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$  be the vectors spanning  $W$ . We then construct an orthonormal basis  $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$  using Gram–Schmidt orthonormalization as follows:

$$\bullet \mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{ so that } \mathbf{q}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\bullet \mathbf{b}_2 = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} - \left( \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} \cdot \mathbf{q}_1 \right) \mathbf{q}_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \text{ so that } \mathbf{q}_2 = \frac{1}{3} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

$$\bullet \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \left( \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \cdot \mathbf{q}_1 \right) \mathbf{q}_1 - \left( \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \cdot \mathbf{q}_2 \right) \mathbf{q}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{9} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 4 \end{bmatrix}, \text{ so that } \mathbf{q}_3 = \frac{1}{\sqrt{18}} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 4 \end{bmatrix}.$$

We have thus found the orthonormal basis  $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{18}} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 4 \end{bmatrix}$  for  $W$ .

(b) The orthogonal projection of  $\mathbf{v} = \begin{bmatrix} 2 \\ 6 \\ -1 \\ 3 \end{bmatrix}$  onto  $W$  is

$$(\mathbf{v} \cdot \mathbf{q}_1) \mathbf{q}_1 + (\mathbf{v} \cdot \mathbf{q}_2) \mathbf{q}_2 + (\mathbf{v} \cdot \mathbf{q}_3) \mathbf{q}_3 = 6 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{5}{9} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \frac{11}{18} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 6 \\ 1/2 \\ 3 \end{bmatrix}.$$

(c) From the first part, we know that  $Q = \begin{bmatrix} 0 & 2/3 & -1/\sqrt{18} \\ 1 & 0 & 0 \\ 0 & 2/3 & -1/\sqrt{18} \\ 0 & 1/3 & 4/\sqrt{18} \end{bmatrix}$ .

$$\text{Hence, } R = Q^T A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2/3 & 0 & 2/3 & 1/3 \\ -1/\sqrt{18} & 0 & -1/\sqrt{18} & 4/\sqrt{18} \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 3 & 5/3 \\ 0 & 0 & 2/\sqrt{18} \end{bmatrix}.$$

(d) Clearly,  $\dim W^\perp = 1$ , so that  $W^\perp$  is spanned by a single vector.

One way to determine vectors  $W^\perp$  is to take any vector  $\mathbf{v}$  (not in  $W$ ) and project  $\mathbf{v}$  onto  $W$ . The error of that projection then is in  $W^\perp$ .

Without extra computation, we can therefore take the error of the projection in the second part of this problem.

$$\text{Indeed, the vector } \begin{bmatrix} 2 \\ 6 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 6 \\ 1/2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \\ -3/2 \\ 0 \end{bmatrix} \text{ is a basis for } W^\perp.$$

□

### Problem 2.

(a) Find the least squares solution to  $\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ .

(b) What is the orthogonal projection of  $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$  onto  $W = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right\}$ ?

(c) Determine the least squares line for the data points  $(-2, 1), (-1, 0), (0, 3), (2, 1)$ .

(d) Determine the projection matrix  $P$  for orthogonally projecting onto  $W$ .

**Solution.** Let  $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 3 \end{bmatrix}$  and  $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ .

(a) We compute  $A^T A = \begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix}$  and  $A^T \mathbf{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ , so the normal equations  $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$  are

$$\begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

Solving, we find that the least squares solution is  $\hat{\mathbf{x}} = \frac{1}{35} \begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 9 \\ 1 \end{bmatrix}$ .

(b) The orthogonal projection of  $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$  onto  $W$  is  $A \hat{\mathbf{x}} = \frac{1}{7} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 9 \\ 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 7 \\ 8 \\ 9 \\ 11 \end{bmatrix}$ .

**Check.** The error  $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix} - \frac{1}{7} \begin{bmatrix} 7 \\ 8 \\ 9 \\ 11 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 0 \\ -8 \\ -6 \\ -4 \end{bmatrix}$  is orthogonal to both  $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} -2 \\ -1 \\ 0 \\ 2 \end{bmatrix}$ .

(c) We need to determine the values  $a, b$  for the least squares line  $y = a + bx$ . The equations  $a + bx_i = y_i$  translate into the system

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad \text{that is, } \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}.$$

We have already computed that the least squares solution to that system is  $\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 9 \\ 1 \end{bmatrix}$ .

Hence, the least squares line is  $y = \frac{9}{7} + \frac{1}{7}x$ .

(d) The projection matrix is  $P = A(A^T A)^{-1} A^T = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \frac{1}{35} \begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 21 & 14 & 7 & -7 \\ 14 & 11 & 8 & 2 \\ 7 & 8 & 9 & 11 \\ -7 & 2 & 11 & 29 \end{bmatrix}$ .

□

**Problem 3.** A scientist tries to find the relation between the mysterious quantities  $x$  and  $y$ .

She measures the following values:

$x$	1	2	3	4
$y$	2	5	9	17

- (a) Our scientist has reason to expect that  $y$  is a linear function of the form  $a + bx$ . Find the best estimate for the coefficients. [“best” in the least squares sense]
- (b) Same question if we suppose that  $y$  is a quadratic function of the form  $a + bx + cx^2$ .

**Solution.**

- (a) If we had  $y = a + bx$  exactly, then we could find  $a, b$  by solving the system

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}}_A \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_y = \underbrace{\begin{bmatrix} 2 \\ 5 \\ 9 \\ 17 \end{bmatrix}}_y.$$

To find the least squares estimate, we solve the normal equations  $A^T A \begin{bmatrix} a \\ b \end{bmatrix} = A^T y$ .

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix} \text{ and } A^T y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 9 \\ 17 \end{bmatrix} = \begin{bmatrix} 33 \\ 107 \end{bmatrix}.$$

We solve  $\begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 33 \\ 107 \end{bmatrix}$  to find  $\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix} \begin{bmatrix} 33 \\ 107 \end{bmatrix} = \begin{bmatrix} -4 \\ 49/10 \end{bmatrix}$ .

Hence,  $a = -4$  and  $b = 4.9$ .

- (b) Again, if we had  $y = a + bx + cx^2$  exactly, then we could find  $a, b, c$  by solving the system

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}}_A \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_y = \underbrace{\begin{bmatrix} 2 \\ 5 \\ 9 \\ 17 \end{bmatrix}}_y.$$

(Now, it becomes a bit painful by hand. Feel free to ask Sage!) The normal equations  $A^T A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^T y$  are:

$$\begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 33 \\ 107 \\ 375 \end{bmatrix}.$$

Solving this system, we find  $a = 2.25$ ,  $b = -1.35$  and  $c = 1.25$ .

□

**Problem 4.**

- (a) What can you say about the determinant of an orthogonal matrix?
- (b) Is it true that  $A^T A$  is always symmetric?
- (c) When is  $A^T A$  a diagonal matrix?

(d) Note that  $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ .

Why is it incorrect that the orthogonal projection of  $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$  onto  $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$  is  $2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ ? Explain!

- (e) For which matrices  $A$  is it true that  $A^{-1} = A^T$ ?

**Solution.**

- (a) If  $Q$  is orthogonal, then  $\det(Q) = \pm 1$ .
- (b) Yes,  $A^T A$  is always symmetric:  $(A^T A)^T = A^T (A^T)^T = A^T A$
- (c)  $A^T A$  is a diagonal matrix if and only if the columns of  $A$  are orthogonal.
- (d) The vectors  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$  are not an orthogonal basis for the span.
- (e)  $A^{-1} = A^T$  if and only if  $A$  is an orthogonal matrix. □

**Problem 5.**

- (a) Set up a linear system to find values for the parameters  $a, b, c$  such that  $y = a + bx + \frac{c}{x}$  best fits some given points  $(x_1, y_1), (x_2, y_2), \dots$
- (b) Set up a linear system to find values for the parameters  $a, b$  such that  $y = (a + bx)e^x$  best fits some given points  $(x_1, y_1), (x_2, y_2), \dots$
- (c) Set up a linear system to find values for the parameters  $a, b, c$  such that  $z = a + bx - c\sqrt{y}$  best fits some given points  $(x_1, y_1, z_1), (x_2, y_2, z_2), \dots$

**Solution.**

- (a) The equations  $a + bx_i + c/x_i = y_i$  translate into the system:

$$\underbrace{\begin{bmatrix} 1 & x_1 & 1/x_1 \\ 1 & x_2 & 1/x_2 \\ 1 & x_3 & 1/x_3 \\ \vdots & \vdots & \vdots \end{bmatrix}}_A \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix}}_{\mathbf{y}}$$

Of course, this is usually inconsistent. To find the best possible  $a, b, c$  we compute a least squares solution of the normal equations:

$$A^T A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^T \mathbf{y}$$

(b) The equations  $(a + bx_i)e^{x_i} = y_i$  translate into the system:

$$\underbrace{\begin{bmatrix} e^{x_1} & x_1 e^{x_1} \\ e^{x_2} & x_2 e^{x_2} \\ e^{x_3} & x_3 e^{x_3} \\ \vdots & \vdots \end{bmatrix}}_A \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix}}_y$$

Of course, this is usually inconsistent. To find the best possible  $a, b$  we compute a least squares solution of the normal equations:

$$A^T A \begin{bmatrix} a \\ b \end{bmatrix} = A^T y$$

(c) The equations  $a + bx_i - c\sqrt{y_i} = z_i$  translate into the system:

$$\underbrace{\begin{bmatrix} 1 & x_1 & -\sqrt{y_1} \\ 1 & x_2 & -\sqrt{y_2} \\ 1 & x_3 & -\sqrt{y_3} \\ \vdots & \vdots & \vdots \end{bmatrix}}_A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \end{bmatrix}}_z$$

Of course, this is usually inconsistent. To find the best possible  $a, b, c$  we compute a least squares solution of the normal equations:

$$A^T A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^T z \quad \square$$

**Problem 6.** Let  $W$  be the space of all solutions to  $x_1 + x_2 + x_3 - x_4 = 0$ .

- (a) Find a basis for  $W$ .
- (b) Find a basis for the orthogonal complement  $W^\perp$ .
- (c) Find  $\mathbf{b}_1$  in  $W$  and  $\mathbf{b}_2$  in  $W^\perp$  such that  $\mathbf{b}_1 + \mathbf{b}_2 = (1, 1, 1, 1)^T$ .

**Solution.** Note that  $W = \text{null}(A)$  for the matrix  $A = [1 \ 1 \ 1 \ -1]$ .

- (a)  $A$  is already in RREF, so we can read off that  $W = \text{null}(A)$  consists of the vectors  $\begin{bmatrix} -s_1 - s_2 + s_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}$ .

Hence, a basis for  $W$  is:  $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

- (b) Recall that the orthogonal complement of  $\text{null}(A)$  is  $\text{row}(A)$ .

Hence, a basis for  $W^\perp$  is:  $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ . (Note how this vector is indeed orthogonal to all basis vectors of  $W$ .)

- (c) Note that this means that  $\mathbf{b}_1$  is the orthogonal projection of  $\mathbf{b} = (1, 1, 1, 1)^T$  onto  $W$ , and  $\mathbf{b}_2$  is the orthogonal projection of  $\mathbf{b}$  onto  $W^\perp$ .

The easiest way to compute these is as follows:

Since  $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$  is an orthogonal basis for  $W^\perp$ , we obtain  $\mathbf{b}_2 = \frac{\mathbf{b}_2 \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ .

Consequently,  $\mathbf{b}_1 = \mathbf{b} - \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$ .

[To check, we can verify that  $\mathbf{b}_1$  is indeed in  $W$  by plugging it into the defining equation.] □

**Problem 7.** Suppose that  $A$  is a  $3 \times 5$  matrix of rank 3.

- For each of the four fundamental subspaces of  $A$ , state which space it is a subspace of.
- What are the dimensions of all four fundamental subspaces?
- Which fundamental subspaces are orthogonal complements of each other?
- For the specific matrix  $A = \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 2 & 4 & 0 & 1 & 3 \\ 3 & 6 & 0 & 1 & 4 \end{bmatrix}$ , compute a basis for each fundamental subspace.
- Observe that  $\text{rank}(A) = 3$ . Then, verify that all your predictions made in the first three parts do in fact hold.

**Solution.**

- $\text{col}(A)$  and  $\text{null}(A^T)$  are subspaces of  $\mathbb{R}^3$ , while  $\text{row}(A)$  and  $\text{null}(A)$  are subspaces of  $\mathbb{R}^5$ .
- $\dim \text{col}(A) = 3$ ,  $\dim \text{row}(A) = 3$ ,  $\dim \text{null}(A) = 5 - 3 = 2$ ,  $\dim \text{null}(A^T) = 3 - 3 = 0$ .
- $\text{col}(A)$  and  $\text{null}(A^T)$  are orthogonal complements of each other.

Also,  $\text{row}(A)$  and  $\text{null}(A)$  are orthogonal complements of each other.

- Gaussian elimination:

$$\begin{aligned} & \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 2 & 4 & 0 & 1 & 3 \\ 3 & 6 & 0 & 1 & 4 \end{bmatrix} \xrightarrow[\begin{smallmatrix} R_3 - 3R_1 \Rightarrow R_3 \\ \rightsquigarrow \end{smallmatrix}]{\begin{smallmatrix} R_2 - 2R_1 \Rightarrow R_2 \\ \rightsquigarrow \end{smallmatrix}} \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 0 & -2 & -5 & -5 \\ 0 & 0 & -3 & -8 & -8 \end{bmatrix} \xrightarrow[\begin{smallmatrix} R_3 - \frac{3}{2}R_2 \Rightarrow R_3 \\ \rightsquigarrow \end{smallmatrix}]{R_3 - \frac{3}{2}R_2 \Rightarrow R_3} \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 0 & -2 & -5 & -5 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\ & \xrightarrow[\begin{smallmatrix} -\frac{1}{2}R_2 \Rightarrow R_2 \\ -2R_3 \Rightarrow R_3 \\ \rightsquigarrow \end{smallmatrix}]{-\frac{1}{2}R_2 \Rightarrow R_2} \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[\begin{smallmatrix} R_1 - 3R_3 \Rightarrow R_1 \\ R_2 - \frac{5}{2}R_3 \Rightarrow R_2 \\ \rightsquigarrow \end{smallmatrix}]{R_1 - 3R_3 \Rightarrow R_1} \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[\begin{smallmatrix} R_1 - R_2 \Rightarrow R_1 \\ \rightsquigarrow \end{smallmatrix}]{R_1 - R_2 \Rightarrow R_1} \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{aligned}$$

Hence, we can read off the bases:

$\text{col}(A)$  has basis  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ .

(Knowing that  $\dim \text{col}(A) = 3$ , so that  $\text{col}(A) = \mathbb{R}^3$ , we could have also just written down the standard basis.)

$\text{row}(A)$  has basis  $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 4 \end{bmatrix}$ .

$\text{null}(A)$  consists of the vectors  $\begin{bmatrix} -2s_1 - s_2 \\ s_1 \\ 0 \\ -s_2 \\ s_2 \end{bmatrix}$  and so has basis  $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ .

$\text{null}(A^T)$  has dimension 0 (contains only the zero vector), and so has an empty basis (consisting of 0 vectors).

- The rank is the number of pivots, which is indeed 3 (also equals  $\dim \text{col}(A)$  and  $\dim \text{row}(A)$ ).

We predicted all the dimensions accurately. □