1 Preparing for Midterm 1

- These problems are taken from the lectures to help you prepare for our upcoming midterm exam. You can find solutions to all of these in the lecture sketches.
- I will also post additional practice problems before the end of the week.

Example 1. Determine a basis for the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$.

Theorem 2. **(Fundamental Theorem of Linear Algebra, Part I)**
Let A be an $m \times n$ matrix of **rank** r.

- $\dim \operatorname{col}(A) = \boxed{}$ (subspace of $\boxed{}$)
- $\dim \operatorname{row}(A) = \boxed{}$ (subspace of $\boxed{}$)
- $\dim \operatorname{null}(A) = \boxed{}$ (subspace of $\boxed{}$)
- $\dim \operatorname{null}(A^T) = \boxed{}$ (subspace of $\boxed{}$)

Example 3. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$. Determine bases for all four fundamental subspaces.

Theorem 4. **(Fundamental Theorem of Linear Algebra, Part II)**

- $\operatorname{null}(A)$ is the orthogonal complement of $\boxed{}$.
- $\operatorname{null}(A^T)$ is the orthogonal complement of $\boxed{}$.

Example 5. Determine bases for all four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 1 \\ 3 & 6 & 0 & 1 \end{bmatrix}.$$

Verify all parts of the Fundamental Theorem.

Theorem 6. $A\boldsymbol{x} = \boldsymbol{b}$ is consistent $\iff$ $\boldsymbol{b}$ is orthogonal to $\boxed{}$

Example 7. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$. For which $\boldsymbol{b}$ does $A\boldsymbol{x} = \boldsymbol{b}$ have a solution?

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Theorem 8. $\hat{\boldsymbol{x}}$ is a least squares solution of $A\boldsymbol{x} = \boldsymbol{b} \iff$ $\boxed{}$

Example 9. Find the least squares solution to $\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Example 10. Find the least squares solution to $\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 5 \\ 0 \\ 5 \\ 10 \end{bmatrix}$.

Example 11. Find the least squares solution to $\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$.

The (orthogonal) projection $\hat{\boldsymbol{b}}$ of $\boldsymbol{b}$ onto $\operatorname{col}(A)$ is $\hat{\boldsymbol{b}} = \boxed{}$.

Example 12. What is the orthogonal projection of $\begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$ onto $\operatorname{span}\left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$?

Example 13.

(a) What is the orthogonal projection of $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ onto $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$?

(b) What is the orthogonal projection of $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ onto $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$?

Example 14. Determine the line that best fits the data points $(2,1), (5,2), (7,3), (8,3)$.

Example 15. Find the line that best fits the points $(-1,5)$, $(0,0)$, $(1,5)$, $(2,10)$.

Example 16. Set up a linear system to find values for the parameters a, b, c that result in the quadratic curve $y = a + bx + cx^2$ that best fits some given points $(x_1, y_1), (x_2, y_2), \ldots$

Example 17. Find values for the parameters a, b, c such that $y = a + bx + cx^2$ best fits the points $(0,2), (1,3), (3,1), (4,1)$. If working by hand, just set up the system.

Example 18. Set up a linear system to find values for the parameters a, b, c such that $z = a + bx + cy$ best fits some given points $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots$

Example 19.

(a) Find the least squares solution to $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \\ 1 & 0 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 3 \\ 10 \\ -10 \\ 3 \end{bmatrix}$.

(b) What is the orthogonal projection of $\begin{bmatrix} 3 \\ 10 \\ -10 \\ 3 \end{bmatrix}$ onto $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ 0 \end{bmatrix}\right\}$?

(c) Find a, b, c so that $y = a + bx + cx^2$ best fits the points $(1,3), (-1,10), (2,-10), (0,3)$.

Example 20. What is the orthogonal projection of $\boldsymbol{v} = (1,2,3)^T$ onto $\boldsymbol{w} = (1,1,1)^T$?

The (orthogonal) projection of $\boldsymbol{v}$ onto $\boldsymbol{w}$ is $\boxed{}$.

Example 21. Are the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ an orthogonal basis for $\mathbb{R}^3$? Is it orthonormal?

If not, normalize the vectors to produce an orthonormal basis.

If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthogonal basis of V, and $\boldsymbol{w}$ is in V, then

$$\boldsymbol{w} = c_1 \boldsymbol{v}_1 + \ldots + c_n \boldsymbol{v}_n \qquad \text{with} \quad c_j = \boxed{}.$$

Example 22. Express $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ in terms of the basis $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

To compute the orthogonal projection $\hat{\boldsymbol{b}}$ of $\boldsymbol{b}$ onto W, we have two options:

- **(using least squares)** Write $W = \operatorname{col}(A)$, where the columns of A are a basis of W.

  Then, $\hat{\boldsymbol{b}} = \boxed{}$

- **(using orthogonality)** If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ is an orthogonal basis of W, then

$$\hat{\boldsymbol{b}} = \boxed{}.$$

Example 23.

(a) Determine the projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

(b) What is the projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $W^\perp$?

$P = \boxed{}$ is the **projection matrix** for projecting onto $\operatorname{col}(A)$.

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Example 24.

(a) Determine the projection matrix P for projecting onto $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

(b) Determine the projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto W using the projection matrix.

Example 25. If P is a projection matrix, then what is P^2?

Example 26.

(a) What is the matrix P for projecting onto $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$?

(b) Using the projection matrix, project $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ onto $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$.

Example 27. Find an orthogonal basis for $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$.

Example 28. Apply Gram–Schmidt orthogonalization to the basis $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Example 29. A matrix A satisfies $A^T A = I$ if and only if ...

Example 30. What can we say about $\det(Q)$ if Q is orthogonal?

(Gram–Schmidt orthonormalization)

Given a basis $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots$ for W, produce an orthonormal basis $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots$ for W.

- $\boldsymbol{q}_1 =$
- $\boldsymbol{q}_2 =$
- $\boldsymbol{q}_3 =$
- $\boldsymbol{q}_4 = \ldots$

Example 31. Find an orthonormal basis for $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}$.

Example 32. Determine the QR decomposition of $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Example 33. Determine the QR decomposition of $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$.

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