Example 197. ($\pi = 4$, cont'd)

We are constructing curves c_n with the property that $c_n \to c$ where c is the circle. This convergence can be understood, for instance, in the same sense $\|c_n - c\| \to 0$ with the norm introduced as we did for functions.

Since $c_n \to c$ we then wanted to conclude that $\text{perimeter}(c_n) \to \text{perimeter}(c)$, leading to $4 \to \pi$.

However, in order to conclude from $x_n \to x$ that $f(x_n) \to f(x)$ we need that f is continuous (at x)!!

The "function" perimeter, however, is not continuous. In words, this means that (as we see in this example) curves can be arbitrarily close, yet have very different arc length.

We can dig a little deeper: as you learned in Calculus II, the arc length of a function $y = f_n(x)$ for $x \in [a, b]$ is

$$\int_a^b \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \int_a^b \sqrt{1 + f_n'(x)^2}\,\mathrm{d}x.$$

Observe that this involves $f_n'(x)$. Try to see why the operator D that sends f to f' is not continuous with respect to the distance induced by the norm

$$\|f\| = \left(\int_a^b f(x)^2 \mathrm{d}x\right)^{1/2}.$$

In words, two functions f and g can be arbitrarily close, yet have very different derivatives f' and g'.

That's a huge issue in **functional analysis**, which is the generalization of linear algebra to infinite dimensional spaces (like the space of all differentiable functions). The linear operators ("matrices") on these spaces frequently fail to be continuous.

11.4 Fourier series

A **Fourier series** for a function $f(x)$ is a series of the form

$$f(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \cdots$$

You may have seen Fourier series in other classes before. Our goal here is to tie them in with what we have learned about orthogonality.

In these other classes, you would have seen formulas for the coefficients a_k and b_k. We will see where those come from.

Observe that the right-hand side combination of cosines and sines is 2π-periodic.

Let us consider (nice) functions on $[0, 2\pi]$.

Or, equivalently, functions that are 2π-periodic.

We know that a natural inner product for that space of functions is

$$\langle f, g \rangle = \int_0^{2\pi} f(t) g(t) \mathrm{d}t.$$

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Example 198. Show that $\cos(x)$ and $\sin(x)$ are orthogonal (in that sense).

Solution. $\langle\cos(x),\sin(x)\rangle=\int_0^{2\pi}\cos(t)\sin(t)\mathrm{d}t=\left[\frac{1}{2}(\sin(t))^2\right]_0^{2\pi}=0$

In fact:

All the functions $1,\cos(x),\sin(x),\cos(2x),\sin(2x),\ldots$ are orthogonal to each other!

Moreover, they form a basis in the sense that every other (nice) function can be written as a (infinite) linear combination of these basis functions.

Example 199. What is the norm of $\cos(x)$?

Solution. $\langle\cos(x),\cos(x)\rangle=\int_0^{2\pi}\cos(t)\cos(t)\mathrm{d}t=\pi$

Why? There's many ways to evaluate this integral. For instance:

- integration by parts
- using a trig identity
- here's a simple way:
  - $\int_0^{2\pi}\cos^2(t)\mathrm{d}t=\int_0^{2\pi}\sin^2(t)\mathrm{d}t$ (cos and sin are just a shift apart)
  - $\cos^2(t)+\sin^2(t)=1$
  - So: $\int_0^{2\pi}\cos^2(t)\mathrm{d}t=\frac{1}{2}\int_0^{2\pi}1\mathrm{d}x=\pi$

Hence, $\cos(x)$ is not normalized. It has norm $\|\cos(x)\|=\sqrt{\pi}$.

Similarly. The same calculation shows that $\cos(kx)$ and $\sin(kx)$ have norm $\sqrt{\pi}$ as well.

Example 200. How do we find, say, b_2?

Solution. Since the functions 1, $\cos(x)$, $\sin(x)$, $\cos(2x)$, $\sin(2x)$, ..., the term $b_2\sin(2x)$ is the orthogonal projection of $f(x)$ onto $\sin(2x)$.

In particular, $b_2=\frac{\langle f(x),\sin(2x)\rangle}{\langle\sin(2x),\sin(2x)\rangle}=\frac{1}{\pi}\int_0^{2\pi}f(t)\sin(2t)\mathrm{d}x.$

In conclusion:

A (nice) $f(x)$ on $[0,2\pi]$ has the Fourier series

$$f(x)=a_0+a_1\cos(x)+b_1\sin(x)+a_2\cos(2x)+b_2\sin(2x)+\cdots$$

where

$$a_k=\frac{\langle f(x),\cos(kx)\rangle}{\langle\cos(kx),\cos(kx)\rangle}=\frac{1}{\pi}\int_0^{2\pi}f(t)\cos(kt)\mathrm{d}t,$$

$$b_k=\frac{\langle f(x),\sin(kx)\rangle}{\langle\sin(kx),\sin(kx)\rangle}=\frac{1}{\pi}\int_0^{2\pi}f(t)\sin(kt)\mathrm{d}t,$$

$$a_0=\frac{\langle f(x),1\rangle}{\langle 1,1\rangle}=\frac{1}{2\pi}\int_0^{2\pi}f(t)\mathrm{d}t.$$

Just for fun and curiousity!

Recall that we introduced the **dimension** of a vector space as the number of vectors in a/any basis. In Calculus, on the other hand, you learn about curves (1-dimensional), surfaces (2-dimensional) and solids (3-dimensional).

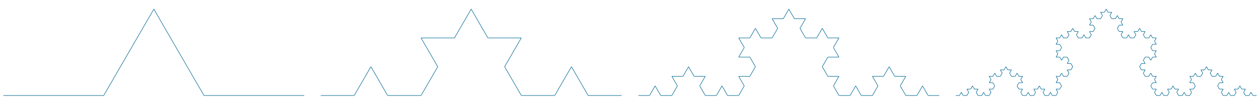
The reason that Linear Algebra is relevant for curved objects like surfaces is that locally these (typically) do look flat (like a plane), so that our tools apply at least locally.

What should a 1.5 dimensional thing look like?

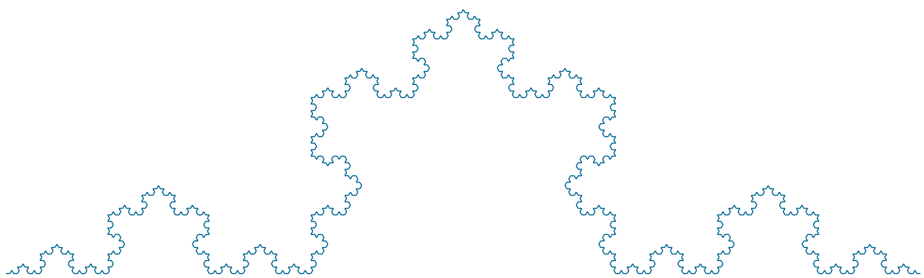
Something between a curve and a surface...

(Note that our linear algebra approach to dimension is not helpful.)

Here is a candidate.



Continuing this process, results in the **Koch snowflake**, a **fractal**:

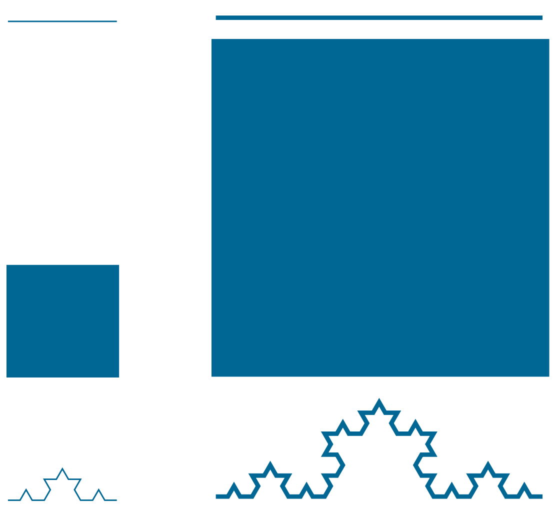


- Its perimeter is infinite!

  Why? At each iteration, the perimeter gets multiplied by $4/3$.

- The table below indicates that its boundary has dimension $\log_3(4) \approx 1.262$!!

the effect of zooming in by a factor of 3



$\times 3$ $\quad d = 1 = \log_3(3)$

$\times 9$ $\quad d = 2 = \log_3(9)$

$\times 4$ $\quad d = \log_3(4) \approx 1.262$

Does this have any practical relevance? Surprisingly, yes!

Have you ever wondered why perimeters of countries are missing from wikipedia? Or, why the coastline of the UK is listed as 11,000 miles by the UK mapping authority but 7,700 miles by the CIA Factbook?

Some of the fun can be found at: https://en.wikipedia.org/wiki/Coastline_paradox