11.3 Linear transformations

Throughout, V and W are vector spaces.

Just like we went from column vectors to abstract vectors (such as polynomials), the concept of a matrix leads to abstract linear transformations.

In the other direction, picking a basis, abstract vectors can be represented as column vectors (see Lecture 35). Correspondingly, linear transformations can then be represented as matrices.

Definition 191. A map $T: V \to W$ is a **linear transformation** if

$$T(c\boldsymbol{x} + d\boldsymbol{y}) = cT(\boldsymbol{x}) + dT(\boldsymbol{y}) \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \text{ in } V \text{ and all } c, d \text{ in } \mathbb{R}.$$

In other words, a linear transformation respects addition and scaling:

- $T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y})$
- $T(c\boldsymbol{x}) = cT(\boldsymbol{x})$

It necessarily sends the zero vector in V to the zero vector in W:

- $T(\boldsymbol{0}) = \boldsymbol{0}$ [because $T(\boldsymbol{0}) = T(0 \cdot \boldsymbol{0}) = 0 \cdot T(\boldsymbol{0}) = \boldsymbol{0}$]

Comment. Linear transformations are special functions and, hence, can be composed. For instance, if $T: V \to W$ and $S: U \to V$ are linear transformations, then $T \circ S$ is a linear transformation $U \to W$ (sending $\boldsymbol{x}$ to $T(S(\boldsymbol{x}))$). If S, T are represented by matrices A, B, then $T \circ S$ is represented by the matrix BA. In other words, matrix multiplication arises as the composition of (linear) functions.

Example 192. The **derivative** you know from Calculus I is linear.

Indeed, the map $D: \left\{ \begin{array}{c} \text{space of all} \\ \text{differentiable} \\ \text{functions} \end{array} \right\} \to \left\{ \begin{array}{c} \text{space of all} \\ \text{functions} \end{array} \right\}$ defined by $f(x) \mapsto f'(x)$ is a linear transformation:

- $\underbrace{D(f(x) + g(x))}_{(f(x)+g(x))'} = \underbrace{D(f(x))}_{f'(x)} + \underbrace{D(g(x))}_{g'(x)}$
- $\underbrace{D(cf(x))}_{(cf(x))'} = \underbrace{cD(f(x))}_{cf'(x)}$

These are among the first properties you learned about the derivative.

Similarly, the **integral** you love from Calculus II is linear:

$$\int_a^b (f(x) + g(x)) \mathrm{d}x = \int_a^b f(x) \mathrm{d}x + \int_a^b g(x) \mathrm{d}x, \quad \int_a^b cf(x) \mathrm{d}x = c \int_a^b f(x) \mathrm{d}x$$

In this form, we are looking at a map $T: \left\{ \begin{array}{c} \text{space of all} \\ \text{continuous} \\ \text{functions} \end{array} \right\} \to \mathbb{R}$ defined by $T(f(x)) = \int_a^b f(x) \mathrm{d}x$.

Example 193. (homework) Consider the space V of all polynomials $p(x)$ of degree 3 or less. The map $D: V \to V$ given by $p(x) \mapsto p'(x)$ is a linear. Write down the matrix M for this linear map with respect to the basis $1, x, x^2, x^3$.

Solution. $M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

For instance, the 3rd column says that x^2 (the 3rd basis element) gets sent to $0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 = 2x$.

Example 194. Consider the map

$$D: \left\{ \begin{matrix} \text{space of poly's} \\ \text{of degree } \leqslant 3 \end{matrix} \right\} \to \left\{ \begin{matrix} \text{space of poly's} \\ \text{of degree } \leqslant 2 \end{matrix} \right\}, \quad p(x) \mapsto p'(x).$$

Write down the matrix M for this linear map with respect to the bases $1, x, x^2, x^3$ and $1, x, x^2$.

Solution. $M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

For instance, the 3rd column says that x^2 (the 3rd basis element) gets sent to $0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 = 2x$.

Example 195. (homework) What is the pseudo-inverse of the matrix M from the previous example. Interpret your finding.

Solution. (final answer only) The pseudo-inverse is $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \end{bmatrix}$.

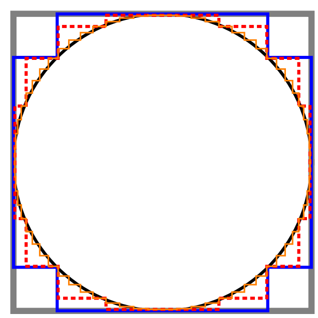
The corresponding linear map sends 1 to x, x to $\frac{1}{2}x^2$ and x^2 to $\frac{1}{3}x^3$. That is, the pseudo-inverse computes the antiderivative of each monomial.

Comment. This is not surprising, since we are familiar from Calculus with the concepts of derivatives and antiderivatives (or integrals), and that these are "pseudo" inverse to each other.

Comment. Similarly, the pseudo-inverse of $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \end{bmatrix}$.

Now, the corresponding linear map sends 1 to x, x to $\frac{1}{2}x^2$, x^2 to $\frac{1}{3}x^3$, and x^3 to 0. That is, the pseudo-inverse computes the antiderivative of each monomial, with the exception of x^3 which gets send to 0 (its antiderivative does not live in the space of polynomials of degree 3).

Example 196. ($\pi = 4$?!) By definition, π is the perimeter of a circle enclosed in a square with edge length 1. The perimeter of the square is 4, which approximates π. To get a better approximation, we "fold" the vertices of the square towards the circle (and get the blue polygon). This construction can be repeated for even better approximations and, in the limit, our shape will converge to the true circle. At each step, the perimeter is 4, so we conclude that $\pi = 4$, contrary to popular belief.



What's going wrong?

Try to put your finger on the mathematical issue!

(It actually is related to the previous examples...)

Armin Straub
straub@southalabama.edu