Comment. Our choice of inner product $\langle f, g\rangle = \int_a^b f(t)g(t)\mathrm{d}t$ for (square-integrable) functions on $[a,b]$ gives rise to the norm $\|f\| = \left(\int_a^b f(t)^2 \mathrm{d}t\right)^{1/2}$. This is known as the L^2-norm (and often written as $\|f\|_2$).

It is the continuous analog of the usual Euclidean norm $\|\boldsymbol{v}\| = (v_1^2 + v_2^2 + \ldots)^{1/2}$ (known as ℓ^2-norm).

There do exist other norms to measure the magnitude of vectors, such as the ℓ_1-norm $\|\boldsymbol{v}\|_1 = |v_1| + |v_2| + \ldots$ or, more generally, for $p \geqslant 1$, the ℓ_p-norms $\|\boldsymbol{v}\|_p = (|v_1|^p + |v_2|^p + \ldots)^{1/p}$.

Likewise, for functions, we have the L^p-norms $\|f\|_p = \left(\int_a^b f(t)^p \mathrm{d}t\right)^{1/p}$.

Only in the case $p = 2$ do these norms come from an inner product. That's a mathematical (as opposed to geometric) reason why we especially care about that case.

Example 188. Find the best approximation of $f(x) = e^x$ on the interval $[0,1]$ using a function of the form $y = ax + b$.

Solution. To find an orthogonal basis for $\operatorname{span}\{1, x\}$, following Gram–Schmidt, we compute

$$x - \begin{pmatrix}\text{projection of}\\ x \text{ onto } 1\end{pmatrix} = x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} 1 = x - \frac{1}{2}.$$

Hence, $1, x - \frac{1}{2}$ is an orthogonal basis for $\operatorname{span}\{1, x\}$.

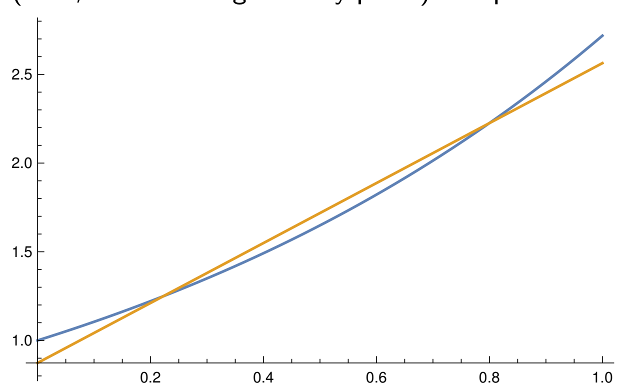
The orthogonal projection of $f\colon [0,1] \to \mathbb{R}$ onto $\operatorname{span}\{1, x\} = \operatorname{span}\left\{1, x - \frac{1}{2}\right\}$ therefore is

$$\begin{aligned}\frac{\langle f, 1\rangle}{\langle 1, 1\rangle} 1 + \frac{\left\langle f, x - \frac{1}{2}\right\rangle}{\left\langle x - \frac{1}{2}, x - \frac{1}{2}\right\rangle}\left(x - \frac{1}{2}\right) &= \int_0^1 f(t)\mathrm{d}t + \frac{\int_0^1 f(t)\left(t - \frac{1}{2}\right)\mathrm{d}t}{\int_0^1 \left(t - \frac{1}{2}\right)^2 \mathrm{d}t}\left(x - \frac{1}{2}\right)\\ &= \int_0^1 f(t)\mathrm{d}t + (12x - 6)\int_0^1 f(t)\left(t - \frac{1}{2}\right)\mathrm{d}t.\end{aligned}$$

In our case, this best approximation is

$$\begin{aligned}\int_0^1 e^t \mathrm{d}t + (12x - 6)\int_0^1 e^t\left(t - \frac{1}{2}\right)\mathrm{d}t &= [e^t]_0^1 + (12x - 6)\left[e^t\left(t - \frac{3}{2}\right)\right]_0^1\\ &= e - 1 + (12x - 6)\left(\frac{3}{2} - \frac{e}{2}\right)\\ &= 4e - 10 + (18 - 6e)x.\end{aligned}$$

(Here, we used integration by parts.) The plot below confirms how good this linear approximation is:



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11.2 Orthogonal polynomials

Let us think about the space of all polynomials (with real coefficients). On that space, we consider the dot product

$$\langle p_1, p_2 \rangle = \int_{-1}^{1} p_1(t) p_2(t) \mathrm{d}t. \qquad (1)$$

Comment. That dot product is useful if we are thinking about the polynomials as functions on $[-1,1]$ only. You can, of course, consider any other interval and you will obtain a shifted version of what we get here.

Example 189. (homework) Are $1, x, x^2, \ldots$ orthogonal (with respect to the inner product (1))?

Solution. Since $\langle x^r, x^s \rangle = \int_{-1}^{1} t^r t^s \mathrm{d}t = \int_{-1}^{1} t^{r+s} \mathrm{d}t$, we find that $\langle x^r, x^s \rangle = \begin{cases} \frac{2}{r+s+1}, & \text{if } r+s \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$

Hence, if $r+s$ is odd, then the monomials x^r and x^s are orthogonal. On the other hand, if $r+s$ is even, then x^r and x^s are not orthogonal.

Example 190. Use Gram-Schmidt to produce an orthogonal basis $\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \ldots$ for the space of polynomials with the dot product (1). Compute $\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4$.

Instead of normalizing these polynomials, **standardize** them so that $\boldsymbol{p}_n(1) = 1$.

Solution. We construct an orthogonal basis $\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \ldots$ from $1, x, x^2, \ldots$ as follows:

- Starting with 1, we find $\boldsymbol{p}_0(x) = 1$.

  For future reference, let us note that $\|\boldsymbol{p}_0\|^2 = \int_{-1}^{1} 1 \mathrm{d}x = 2$.

- Starting with x, Gram–Schmidt produces $x - \begin{pmatrix} \text{projection of} \\ x \text{ onto } \boldsymbol{p}_0 \end{pmatrix} = x - \frac{\langle x, \boldsymbol{p}_0 \rangle}{\langle \boldsymbol{p}_0, \boldsymbol{p}_0 \rangle} \boldsymbol{p}_0 = x - \int_{-1}^{1} t \mathrm{d}t = x$.

  Again, that's already standardized, so that $\boldsymbol{p}_1(x) = x$.

  Comment. The previous problem already told us that x is orthogonal to 1.

  For future reference, let us note that $\|\boldsymbol{p}_1\|^2 = \int_{-1}^{1} t^2 \mathrm{d}t = \frac{2}{3}$.

- Starting with x, Gram–Schmidt produces $x^2 - \begin{pmatrix} \text{projection of } x^2 \\ \text{onto span}\{\boldsymbol{p}_0, \boldsymbol{p}_1\} \end{pmatrix} = x^2 - \frac{\langle x^2, \boldsymbol{p}_0 \rangle}{\langle \boldsymbol{p}_0, \boldsymbol{p}_0 \rangle} \boldsymbol{p}_0 - \frac{\langle x^2, \boldsymbol{p}_1 \rangle}{\langle \boldsymbol{p}_1, \boldsymbol{p}_1 \rangle} \boldsymbol{p}_1$ $= x^2 - \frac{1}{2} \int_{-1}^{1} t^2 \mathrm{d}t - \frac{x}{2/3} \int_{-1}^{1} t^3 \mathrm{d}t = x^2 - \frac{1}{3}$.

  Hence, standardizing, $\boldsymbol{p}_2(x) = \frac{1}{2}(3x^2 - 1)$.

  Comment. The previous problem told us that x^2 is orthogonal to x (but not to 1).

- **(homework)** Continuing, we find $\boldsymbol{p}_3(x) = \frac{1}{2}(5x^3 - 3x)$ and $\boldsymbol{p}_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$.

Comment. These famous polynomials are known as the **Legendre polynomials**. The Legendre polynomial $\boldsymbol{p}_n$ is an even function if n is even, and an odd function if n is odd (can you explain why?!).

An explicit formula is $\boldsymbol{p}_n(x) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k}^2 (x+1)^k (x-1)^{n-k}$.

For instance, $\boldsymbol{p}_2(x) = \frac{1}{4}((x-1)^2 + 2^2(x-1)(x+1) + (x+1)^2) = \frac{1}{2}(3x^2 - 1)$

https://en.wikipedia.org/wiki/Legendre_polynomials

Comment. Legendre polynomials are an example of **orthogonal polynomials**. Each choice of dot product gives rise to a family of such orthogonal polynomials.

https://en.wikipedia.org/wiki/Orthogonal_polynomials

Comment. It is also particularly natural to consider the dot product (1), where the integral is from 0 to 1. In that case, we obtain what's known as the shifted Legendre polynomials $\tilde{\boldsymbol{p}}_n(x) = \boldsymbol{p}_n(2x-1)$.

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