Example 180. Give a basis for the space of all polynomials of degree 3 or less.

Solution. The polynomials $1, x, x^2, x^3$ form a basis for that space.

To see why, recall that the basis vectors need to do two things: they need to span the space and they need to be independent. Equivalently, every element in the space needs to representable as a linear combination of the basis elements (they span) and this representation must be unique (they are independent).

We are familiar with the fact that every polynomial $p(x)$ of degree 3 or less can be written as $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, and that the coefficients a_0, a_1, a_2, a_3 are unique.

Important observation. As a consequence, the $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ can be expressed as $\begin{bmatrix} a_o \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

That is, we can work with the familiar column vectors in any vector space, as soon as we have picked a basis. The next example will illustrate how to translate back and forth.

Example 181. Give a basis for the space of all polynomials $p(x)$ of degree 2 or less such that $p(3) = 0$.

Solution. (calculus) From Calculus, we know that $p(3) = 0$ means that 3 is a root of the polynomial, and that, as a consequence, the polynomial factors as $p(x) = (x-3)q(x)$, where $q(x)$ is another polynomial.

Hence, a basis for our space is $x - 3, x(x-3)$.

[That is, we are multiplying $x - 3$ with $1, x, x^2 \ldots$ but stop at x because we are restricted to degree 2 or less.]

Solution. (linear algebra) Let us start with the basis $1, x, x^2$ for the space of all polynomials $p(x)$ of degree 2 or less.

Then, we can identify the polynomial $p(x) = a_0 + a_1x + a_2x^2$ with the vector $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$.

The condition $p(3) = 0$ translates into $a_0 + 3a_1 + 9a_2 = 0$.

In other words, the space of polynomials $p(x)$ of degree 2 or less such that $p(3) = 0$ translates into $\operatorname{null}(\begin{bmatrix} 1 & 3 & 9 \end{bmatrix})$.

A basis for $\operatorname{null}(\begin{bmatrix} 1 & 3 & 9 \end{bmatrix})$ is $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \\ 1 \end{bmatrix}$.

The corresponding polynomials are $-3 + x$ and $-9 + x^2$.

Example 182. (homework) Give a basis for the space of all polynomials $p(x)$ of degree 3 or less such that $p(1) = 0$ and $p'(1) = 0$.

Solution. Let us start with the basis $1, x, x^2, x^3$ for the space of all polynomials $p(x)$ of degree 3 or less.

Then, we can identify the polynomial $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ with the vector $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

The condition $p(1) = 0$ translates into $a_0 + a_1 + a_2 + a_3 = 0$.

Since $p'(x) = a_1 + 2a_2x + 3a_3x^2$, the condition $p'(1) = 0$ translates into $a_1 + 2a_2 + 3a_3 = 0$.

In other words, the space of all polynomials $p(x)$ of degree 3 or less such that $p(1) = 0$ and $p'(1) = 0$ translates into $\operatorname{null}\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}\right)$.

A basis for $\operatorname{null}\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}\right)$ is $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$. (Fill in the details!)

The corresponding polynomials are $1 - 2x + x^2$ and $2 - 3x + x^3$.

[Check that they indeed satisfy $p(1) = 0$ and $p'(1) = 0$.]

Comment. Let's note that it was to be expected from the beginning that the space is 2-dimensional. The space of all polynomials $p(x)$ of degree 3 or less has dimension 4. Since we impose 2 (independent) conditions, the dimension of our space is $4 - 2 = 2$.

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Example 183. Give a basis for the space of all polynomials.

Solution. $1, x, x^2, x^3, \ldots$

Indeed, every polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ can be written uniquely as a sum of these basis elements. ("can be" = span; "uniquely" = independent)

Comment. The dimension is ∞. But we can make a list of basis elements, which is the "smallest kind of ∞" and is referred to as **countably infinite**. For the space of all functions, no such list can be made.

Just for fun. Let us indicate this difference in infiniteness in a slightly simpler situation: first, the natural numbers $0, 1, 2, 3\ldots$ are infinite but they are countable, because we can make a (infinite but complete) list starting with a first, then a second element and so on (hence, the name "countable"). On the other hand, consider the real numbers between 0 and 1. Clearly, there is infinitely many such numbers. The somewhat shocking fact (first realized by Georg Cantor in 1874) is that every attempt of making a complete list of these numbers must fail because every list will inevitably miss some numbers. Here's a brief indication of how the famous diagonal argument goes: suppose you can make a list, say:

$$\begin{array}{ll} \#1 & 0.111111\ldots \\ \#2 & 0.123456\ldots \\ \#3 & 0.750000\ldots \\ \vdots & \end{array}$$

Now, we are going to construct a new number $x = 0.x_1 x_2 x_3 \ldots$ with decimal digits x_i in such a way that the digit x_i differs from the ith digit of number $\#i$ on our list. For instance, $0.231\ldots$ in our case (for instance, $x_3 = 1$ differs from 0, the 3rd digit of sequence $\#3$). By construction, the number x is missing from the list.

Follow-up. What if we only consider rational numbers in the interval $[0, 1]$? Does the previous argument still apply? Or, can we now make a list?

Comment on fun. The statement "some infinities are bigger than others" nicely captures our observation. It appears in the book *The Fault in Our Stars* by John Green, where it is said by a cranky old author who attributes it to Cantor. Hazel, the main character, later reflects on that statement and compares $[0, 1]$ to $[0, 2]$. Can you explain why that is actually not what Cantor meant$\ldots$?