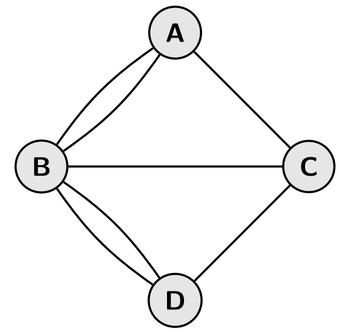
Example 174. We discussed the famous problem of the **Seven Bridges of Königsberg**.

https://en.wikipedia.org/wiki/Seven_Bridges_of_Königsberg

Historically, Euler's clever (and general!!) solution (1736) laid the foundation of graph theory and topology.

A key insight is to focus on the 4 landmasses instead of the 7 bridges. Let's call these landmasses A, B, C, D. They are connected via the bridges as in the graph to the right.



The problem at hand is to traverse the graph via a sequence of consecutive edges such that each edge is traversed exactly once.

Obviously (think about it!), if a (connected) graph is thus traversable, then

- either all nodes have even degree (the **degree** refers to the number of edges at that node), or
- two nodes (start and end of our route) have odd degree and all other nodes have even degree.

Neither of these conditions is true for our graph. Hence the 7 bridges problem has no solution.

More is true! However, more is true and this makes Euler's approach much more interesting. Namely, the two conditions above are true **if and only if** a (connected) graph is traversable.

Can you prove that claim? That is, for a graph satisfying the two conditions, you need to show that it can indeed be traversed. One way to do that is by induction on the number of edges (roughly: start with an edge attached to a node with odd degree (if not present, just take any edge); then, remove that edge and keep it in mind as the first step of your journey...).

10 Linear approximations to arbitrary functions

- **(Calculus 1)** Let $f(x)$ be any (nice; i.e. differentiable) function $\mathbb{R}\to\mathbb{R}$.

  Suppose the best linear approximation to $f(x)$ at $x=a$ is $f(x)\approx f(a)+m(x-a)$.

  Geometrically, $x\mapsto f(a)+m(x-a)$ describes the tangent line to $f(x)$ at $x=a$.

  In Calculus 1, you learn that the crucial quantity m gives rise to the **derivative**, $m=f'(a)$, and you learn how to compute it, given a function $f(x)$.

- **(Calculus 3)** Let $\boldsymbol{f}(\boldsymbol{x})$ be any (nice) function $\mathbb{R}^n\to\mathbb{R}^m$.

  Now, the best linear approximation to $\boldsymbol{f}(\boldsymbol{x})$ at $\boldsymbol{x}=\boldsymbol{a}$ is $\boldsymbol{f}(\boldsymbol{x})\approx \boldsymbol{f}(\boldsymbol{a})+M(\boldsymbol{x}-\boldsymbol{a})$.

  Here, M is a $m\times n$ matrix. (Why these dimensions?!)

  In Calculus 3, you again learn how to compute the quantity $M=D\boldsymbol{f}(\boldsymbol{a})$ (the **derivative** or **Jacobian matrix**), given a function $f(x)$. Indeed:

$$\boldsymbol{f}(\boldsymbol{x})=\begin{bmatrix} f_1(x_1,\ldots,x_n)\\ \vdots \\ f_m(x_1,\ldots,x_n)\end{bmatrix},\quad D\boldsymbol{f}=\begin{bmatrix} \frac{\partial}{\partial x_1}f_1 & \cdots & \frac{\partial}{\partial x_n}f_1 \\ \vdots & & \vdots \\ \frac{\partial}{\partial x_1}f_m & \cdots & \frac{\partial}{\partial x_n}f_m\end{bmatrix}$$

Comment. One important point of this discussion is that, through our linear algebra glasses, the transition from Calculus 1 to Calculus 3 is exactly as expected.

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Example 175. Determine the best linear approximation to $\boldsymbol{f}(x,y)=\begin{bmatrix} x^2+y^2 \\ xy+1 \end{bmatrix}$ at $\boldsymbol{x}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Solution. The Jacobian matrix at $\boldsymbol{x}=\begin{bmatrix} x \\ y \end{bmatrix}$ is

$$D\boldsymbol{f}(x,y)=\begin{bmatrix} \frac{\partial}{\partial x}(x^2+y^2) & \frac{\partial}{\partial y}(x^2+y^2) \\ \frac{\partial}{\partial x}(xy+1) & \frac{\partial}{\partial x}(xy+1) \end{bmatrix}=\begin{bmatrix} 2x & 2y \\ y & x \end{bmatrix}, \quad D\boldsymbol{f}(2,1)=\begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix}.$$

Hence, the best linear approximation to $\boldsymbol{f}(x,y)$ at $\boldsymbol{x}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is

$$\boldsymbol{f}(x,y)\approx\boldsymbol{f}(2,1)+\begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix}-\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)=\begin{bmatrix} 5 \\ 3 \end{bmatrix}+\begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x-2 \\ y-1 \end{bmatrix}.$$

Comment. For most purposes, this is the nicest form to write the linear approximation. If we want to multiply out the matrix-vector product, we get

$$\boldsymbol{f}(x,y)\approx\begin{bmatrix} 4x+2y-5 \\ x+2y-1 \end{bmatrix}.$$

Note how we solved two independent problems at once: approximating x^2+y^2 at $(x,y)=(2,1)$, and approximating $xy+1$ at $(x,y)=(2,1)$.

Example 176. (homework) For $\boldsymbol{f}(r,\theta)=\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$, determine the Jacobian matrix (i.e. the derivative) and its determinant.

Solution. The Jacobian matrix is $\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$.

Its determinant is $\det\left(\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}\right)=r\cos^2\theta+r\sin^2\theta=r$.

Example 177. If you have taken Calculus 3, you have learned about **substitution** in multiple integrals. To make the change of variables $\boldsymbol{x}=\boldsymbol{g}(\boldsymbol{u})$, that is, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} g_1(u_1,u_2) \\ g_2(u_1,u_2) \end{bmatrix}$, we have

$$\int\int_R f(\boldsymbol{x})\mathrm{d}x_2\mathrm{d}x_1=\int\int_G f(\boldsymbol{g}(\boldsymbol{u}))|D\boldsymbol{g}(\boldsymbol{u})|\mathrm{d}u_2\mathrm{d}u_1$$

- where G is the region in the u_1u_2-plane corresponding to the region R in the x_1x_2-plane,
- and $|D\boldsymbol{g}(\boldsymbol{u})|=\det\begin{bmatrix} \frac{\partial}{\partial u_1}g_1 & \frac{\partial}{\partial u_2}g_1 \\ \frac{\partial}{\partial u_1}g_2 & \frac{\partial}{\partial u_2}g_2 \end{bmatrix}$ is the **Jacobian determinant**.

Comment. Have another look at Example 84. In that example, we observed that $\det(A)$ measures by much a little volume (or area) is scaled under $\boldsymbol{x}\mapsto A\boldsymbol{x}$. Hence, it makes perfect sense that the formula for substitution needs to take into account how much the function $\boldsymbol{g}$ changes volumes. Locally, $\boldsymbol{u}\mapsto\boldsymbol{g}(\boldsymbol{u})$ is approximated (up to a shift) by multiplication with $D\boldsymbol{g}$, and so the change in volume is measured by the Jacobian determinant.

For instance. From your computation in the previous example, it follows that, using **polar coordinates** $x=r\cos\theta$, $y=r\sin\theta$, we have the familiar formula

$$\int\int_R f(x,y)\mathrm{d}y\mathrm{d}x=\int\int_G f(r\cos\theta,r\sin\theta)r\mathrm{d}r\mathrm{d}\theta.$$