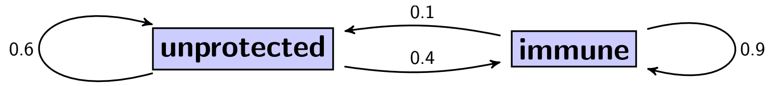
Sketch of Lecture 31 Fri, 4/7/2017

Example 168. Consider a fixed population of people with or without active immunization against some disease (like tetanus). Suppose that, each year, 40% of those unprotected get vaccinated while 10% of those with immunization lose their protection.
What is the immunization rate in the long run? (The long term equilibrium.)

Solution.



x_t: proportion of population unprotected at time t (in years)
y_t: proportion of population immune at time t [Note that $x_t + y_t = 1$.]

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0.6x_t + 0.1y_t \\ 0.4x_t + 0.9y_t \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

The matrix $\begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$ is the **transition matrix** of this dynamical system, because it describes the transition from time t to time $t+1$. This particular one is a **Markov matrix** (or stochastic matrix): its columns add to 1 and it has no negative entries.

$\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an equilibrium if $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$. In other words, $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ is an eigenvector with eigenvalue 1.

The 1-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -0.4 & 0.1 \\ 0.4 & -0.1 \end{bmatrix} \right)$, which has basis $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Since $x_\infty + y_\infty = 1$, we conclude that $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \frac{1}{1+4} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 4/5 \end{bmatrix}$.

Hence, the immunization rate in the long term equilibrium is $4/5 = 80\%$.
[Ponder about why this is a reasonable value!]

Comment. Our flow diagram above can be thought of as a directed graph with two nodes and four edges (including two edges which connect a node to itself). Note that the transition matrix is very similar to the adjacency matrix of that graph (if we take its transpose).

Comment. What's the other eigenvalue of the transition matrix? No need to compute the characteristic polynomial: we can easily see that it is $0.5 = 0.6 \cdot 0.9 - 0.1 \cdot 0.4$ because the product of the eigenvalues equals the determinant!

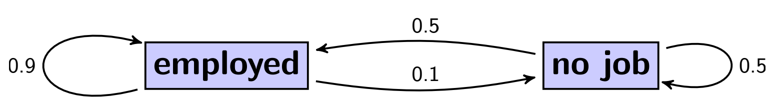
The 0.5-eigenspace is spanned by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Comment. Will the immunization rate always stabilize and approach the long term equilibrium? Yes! This is a consequence of the other eigenvalue of the transition matrix satisfying $|0.5| < 1$. If we start in state $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 4 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = 1^n \cdot a \begin{bmatrix} 1 \\ 4 \end{bmatrix} + (0.5)^n \cdot b \begin{bmatrix} -1 \\ 1 \end{bmatrix} \xrightarrow[\text{as } n \to \infty]{} a \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Random comment. A rule of thumb is that tetanus vaccination begins to wear off after about 10 years (somewhat in line with the 0.1 transition proportion in this example). However, the tetanus immunzation rate in the United States appears to be considerable less than the 80% we found in this (awfully simplistic) example.
https://www.cdc.gov/mmwr/preview/mmwrhtml/mm5940a3.htm

Example 169. (homework) Consider a fixed population of people with or without a job. Suppose that, each year, 50% of those unemployed find a job while 10% of those employed lose their job. What is the unemployment rate in the long term equilibrium?

Solution.



Proceeding, as in the previous example, the transition matrix is $\begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}$.

In the end, we find that the unemployment rate in the long term equilibrium is $1/6 \approx 16.7\%$.

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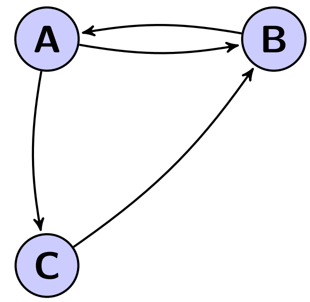
Example 170. Suppose the internet consists of only the three webpages A, B, C.

We wish to rank these webpages in order of "importance".



The idea. Instead of analyzing each webpage (which would be a lot of work!) we will try to only use the information how the pages are linked to each other. The idea being that an "important" page should be linked to from many other pages.

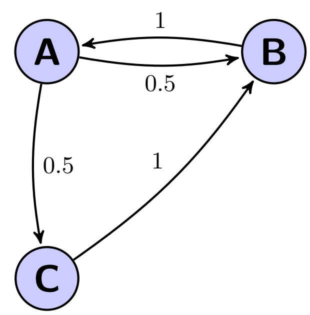
A and B have a link to each other. Also, A links to C and C links to B. If you keep randomly clicking from one webpage to the next, what proportion of the time will you be at each page?

The idea. We will assign ranking to those pages according to how frequently such a random surfer would visit these pages.

Comment. Before we start computing, stop for a moment, and think about how you would rank the webpages.

Solution. Let a_t be the probability that we will be on page A at time t. Likewise, b_t, c_t are the probabilities that we will be on page B or C.

The transition from one state to the next now works exactly as in the previous example. We get the following transition matrix:



$$\begin{bmatrix} a_{t+1} \\ b_{t+1} \\ c_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \cdot a_t + 1 \cdot b_t + 0 \cdot c_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 1 \cdot c_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 0 \cdot c_t \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} a_t \\ b_t \\ c_t \end{bmatrix}$$

To find the equilibrium state, we again determine an appropriate 1-eigenvector.

The 1-eigenspace is $\text{null}\left(\begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \\ \frac{1}{2} & 0 & -1 \end{bmatrix}\right)$ which has basis $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

The corresponding equilibrium state is $\frac{1}{5}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. In this context, this is also known as the **PageRank vector**.

In other words, after browsing randomly for a long time, there is (about) a $\frac{2}{5} = 40\%$ chance to be at page A, a $\frac{2}{5} = 40\%$ chance to be at page B, and a $\frac{1}{5} = 20\%$ chance to be at page C.

We therefore rank A and B highest (tied), and C lowest.

Just checking. Maybe we were expecting B to be ranked above A, because B is the only page that has two incoming links. However, if we are at page B, then our next click will be to page A, which is why A and B receive equal ranking.

This method of ranking is the famous **PageRank** algorithm (underlying Google's search algorithm).

By the way, the algorithm is named, not after ranking web"pages", but after Larry Page (who founded Google in 1998 together with Sergey Brin).

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