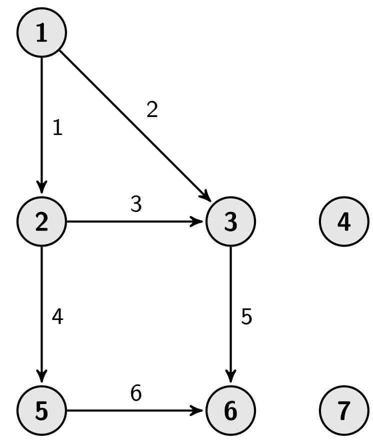
Sketch of Lecture 30

Wed, 4/5/2017

Example 163. Consider the graph to the right. Determine its edge-node incidence matrix M.



(a) What are $\dim \operatorname{null}(M)$ and $\dim \operatorname{null}(M^T)$?

(b) Give bases for $\operatorname{null}(M)$ and $\operatorname{null}(M^T)$.

Solution. The edge-node incidence matrix is

$$M = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

(a) The graph has 3 connected components. Hence, $\dim \operatorname{null}(M) = 3$.
The graph has 2 independent loops. Hence, $\dim \operatorname{null}(M^T) = 2$.

(b) A basis for $\operatorname{null}(M)$ is: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (one vector for each connected component)

A basis for $\operatorname{null}(M^T)$ is: $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ (one vector for the loops $1 \to 2 \to 3$ and $2 \to 5 \to 6 \to 3$)

Definition 164. Let G be a (directed) graph n nodes.
The **adjacency matrix** of G is the $n \times n$ matrix A with

$$A_{i,j} = \begin{cases} 1, & \text{if there is an edge from node } i \text{ to node } j, \\ 0, & \text{otherwise.} \end{cases}$$

Comment. In the case of an undirected graph, each edge is treated as being directed both ways. Consequently, the entries $A_{i,j}$ and $A_{j,i}$ are the same, so that the matrix A is symmetric.

Example 165. Determine the adjacency matrix for the graph in Example 163.

Solution. Since there is 7 nodes, the adjacency matrix A is a 7×7 matrix. Since there are 6 edges, this matrix has 6 entries which are 1 (and all other entries are 0).

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If A is the adjacency matrix of a graph, then the power A^n has an interesting interpretation:

$$(A^n)_{i,j} = \#\text{ of paths from node } i \text{ to node } j \text{ along exactly } n \text{ edges}$$

Why? Can you see why this is true for A^2? (Then, the general statement follows inductively.)

Recall that matrix multiplication works in such a way that the entry $(A^2)_{i,j}$ is computed as the dot product of the ith row of A with the jth column of A.

The ith row of A encodes edges from node i, whereas the jth column of A encodes edges into node j. Hence, the dot product gives all possibilities going from node i via an edge to some node k, and then from k via another edge to node j. (This takes some pondering!)

Example 166. Compute A^2 for the adjacency matrix from the previous example. Compare with the interpretation as numbers of 2-edge paths.

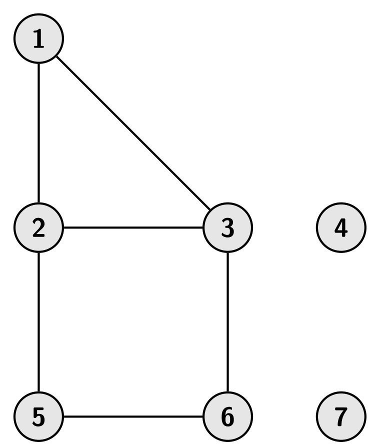
Solution.

$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The entry $(A^2)_{1,3}$ is 1 because there is 1 path along 2 edges from node 1 to node 3: $1 \to 2 \to 3$ (namely the path via node 2). On the other hand, $(A^2)_{3,1} = 0$ because there is no 2-edge path from node 3 to node 1.

$(A^2)_{2,6} = 2$ because there is 2 paths along 2 edges from node 2 to node 6: $2 \to 3 \to 6$, $2 \to 5 \to 6$.

Example 167. (homework) Determine the adjacency matrix A for the graph to the right. Then compute A^2 and compare with the interpretation as numbers of 2-edge paths. What is $(A^3)_{1,1}$?



Solution. The adjacency matrix is

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 1 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Comment. Because this is an undirected graph, each of the six edges gives rise to two entries which are 1.

$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 1 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 1 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 0 & 1 & 1 & 0\\ 1 & 3 & 1 & 0 & 0 & 2 & 0\\ 1 & 1 & 3 & 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 2 & 0 & 2 & 0 & 0\\ 1 & 2 & 0 & 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For instance, $(A^2)_{3,5} = 2$ because there is 2 paths along 2 edges from node 3 to node 5: $3 \to 2 \to 5$, $3 \to 6 \to 5$.

Likewise, $(A^2)_{2,2} = 3$ because there is 3 paths along 2 edges from node 2 to node 2: $2 \to 1 \to 2$, $2 \to 5 \to 2$, $2 \to 3 \to 2$.

Finally, $(A^3)_{1,1} = 2$ because the 3-edge paths from node 1 to node 1 are $1 \to 2 \to 3 \to 1$ and $1 \to 3 \to 2 \to 1$.