Applications of (directed) graphs are numerous and only limited by your imagination!

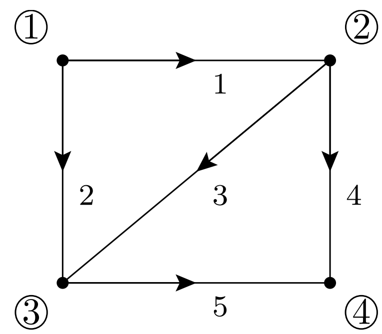
results of (sport) tournaments, webpages linking to each other (PageRank algorithm!), ...

9.2 Meaning of the left null space

The $\boldsymbol{y}$ in $A^T\boldsymbol{y}$ is assigning values to each edge.

You may think of assigning **currents** to each edge. In our running example:

$$\underbrace{\begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}}_{A^T} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} -y_1 - y_2 \\ y_1 - y_3 - y_4 \\ y_2 + y_3 - y_5 \\ y_4 + y_5 \end{bmatrix}$$



Crucial observation. $A^T\boldsymbol{y} = \boldsymbol{0} \iff \boldsymbol{y}$ assigns values to edges in such a way that, at each node, the sum of (oriented) edge values is zero

Comment. When thinking of currents, this is **Kirchhoff's first law**. (At each node, incoming and outgoing currents balance.)

What is the simplest way to balance current? Assign the current in a **loop**!

Example. In our running example, for instance, $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ are in $\operatorname{null}(A^T)$. Check!

Example 159. Compute a basis for $\operatorname{null}(A^T)$. Interpret your answer in terms of loops.

Solution. To solve $A^T\boldsymbol{y}$ for our graph, we perform the usual Gaussian elimination:

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can then simply read off the general solution as $\begin{bmatrix} s_1 - s_2 \\ -s_1 + s_2 \\ s_1 \\ -s_2 \\ s_2 \end{bmatrix} = s_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

Hence, a basis for $\operatorname{null}(A^T)$ is $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

Both of these two basis vectors correspond to loops. (Make sure you see that!)

Note that we get the "simpler" loop $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ as the combination $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

(left nullspace of edge-node incidence matrix)

$\dim \operatorname{null}(A^T)$ is the number of (independent) loops.

For large graphs, computing a basis for $\operatorname{null}(A^T)$ gives a nice way to computationally find all loops.

Example 160. (homework) For the graph from Example 158 and its edge-node incidence matrix A, what is $\dim\operatorname{null}(A^T)$?

Solution. That graph obviously has no loops. Hence, $\dim\operatorname{null}(A^T) = 0$.

Alternatively, note that $A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$ clearly has $\operatorname{rank}(A) = 2$, so that $\dim\operatorname{null}(A^T) = 2 - 2 = 0$.

Comment. Keep in mind that these small examples are somewhat misleading: here, we can just "see" how many loops there are (none) and so we conclude that $\dim\operatorname{null}(A^T) = 0$. For a large graph, the situtation would be the reverse: we cannot just look at a graph with hundreds or millions of node to see how many loops (or connected components) it has; however, we can certainly just compute $\dim\operatorname{null}(A^T)$ (or $\dim\operatorname{null}(A)$) to find out.

9.3 Euler's formula

Theorem 161. (Euler's formula) In any connected graph,

$$\#\text{nodes} \; - \; \#\text{edges} \; + \; \#\text{loops} \; = 1.$$

Example. In our running example, for instance, $4 - 5 + 2 = 1$.

Proof. Let A be the $m \times n$ edge-node incidence matrix of the graph.

- $\#\text{edges} = m$
- $\#\text{nodes} = n$
- $\#\text{loops} = \dim\operatorname{null}(A^T)$
- $1 = \dim\operatorname{null}(A)$

The FTLA states that, if $r = \operatorname{rank}(A)$,

- $\dim\operatorname{null}(A) = n - r$ (which means $n - r = 1$, that is, $r = n - 1$)
- $\dim\operatorname{null}(A^T) = m - r = m - n + 1$

Hence,

$$\#\text{nodes} \; - \; \#\text{edges} \; + \; \#\text{loops} \; = n - m + (m - n + 1) = 1, \qquad \square$$

Comment. From the proof (or by applying Euler's formula to each connected component of the graph), we see that, for any graph,

$$\#\text{nodes} \; - \; \#\text{edges} \; + \; \#\text{loops} \; = \#\text{connected components}.$$

Example 162. Euler's formula generalizes to other settings than directed graphs. For instance, you might have seen the following version of Euler's formula:

In a convex polyedron,

$$\#\text{vertices} \; - \; \#\text{edges} \; + \; \#\text{faces} \; = 2.$$

Why? Note that we can convert a convex polyhedron to a (randomly directed) graph by only keeping the nodes and edges. Then, each face corresponds to a loop. However, these loops are not quite independent. Can you see how the "last" face loop is a combination of the previous ones? In the end, $\#\text{faces} = \#\text{loops} + 1$. That's why the right-hand side increases from 1 to 2.