8 Linear differential equations

Example 128. (warmup) Solve the **differential equation** (DE) $y' = 2$.

Solution. From Calculus, we know that the solutions are of the form $y(t) = 2t + C$.

Comment. To get a unique solution, we need to specify additional information, like an initial condition.

Example 129. (warmup) Solve the **initial value problem** (IVP) $y' = 2$, $y(0) = 1$.

Solution. This has the unique solution $y(t) = 2t + 1$.

Example 130. Which functions $y(t)$ satisfy the differential equation $y' = y$?

Solution. $y(t) = e^t$ and, more generally, $y(t) = Ce^t$. (And nothing else.)

Recall from Calculus the Taylor series $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$

Example 131. Show that the differential equation $y' = 3y$ is solved by $y(t) = Ce^{3t}$.

Solution. Indeed, if $y(t) = Ce^{3t}$, then $y'(t) = 3Ce^{3t} = 3y(t)$.

Comment. It is important to realize that we can always easily check whether a function solves a differential equation. This means that you can use computer algebra systems like Sage to solve differential equations without trust issues (because you might be unfamiliar with the techniques for solving).

Example 132. Solve the differential equation $y' = ay$ with initial condition $y(0) = y_0$.

Solution. As in the previous example, the general solution to $y' = ay$ is $y(t) = Ce^{at}$.

Since $y(0) = Ce^0 = C = y_0$, we conclude that the unique solution to the IVP is $y(t) = e^{at} y_0$.

Comment. It looks silly to write $e^{at} y_0$ instead of $y_0 e^{at}$ here, but we will soon replace the number a with a matrix A, and in that case only $e^{At} y_0$ makes sense.

Example 133. We will only discuss linear DEs. Non-linear differential equations include $y' = y^2 + 1$ or the second-order equation $y'' = \sin(ty') + y$.

The order of a DE indicates the highest occuring derivative.

Note, however, that $y'' = \sin(t) y' + y$ is a linear DE, because y and its derivatives occur linearly.

We will see here how to solve those linear DEs which have constant coefficients. That is, the coefficients of y are constants, as opposed to functions (like $\sin(t)$) depending on t.

Example 134. Our goal is to solve (systems of) differential equations like:

$$\begin{array}{rcrrr} y_1' &=& 2y_1 & & \\ y_2' &=& -y_1 & +3y_2 & +y_3 \\ y_3' &=& -y_1 & +y_2 & +3y_3 \end{array} \qquad \begin{array}{rcl} y_1(0) &=& 1 \\ y_2(0) &=& 0 \\ y_3(0) &=& 2 \end{array}$$

In matrix form, this becomes

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

The key idea will be to solve $\boldsymbol{y}' = A\boldsymbol{y}$ by introducing e^{At}.

Example 135. (homework) Diagonalize $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.

Solution. (final solution only) $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 4 \end{bmatrix}$.

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Example 136. Write the (second-order) differential equation $y'' = 2y' + y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$ and $y_2 = y'$. Then $y'' = 2y' + y$ becomes $y_2' = 2y_2 + y_1$.

Then $y'' = 2y' + y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_1 + 2y_2 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \boldsymbol{y}$.

Comment. This is further indication why we would care about systems of differential equations, even if we work with just one function.

Example 137. (homework) Write $y''' = 3y'' - 2y' + y$ in the form $\boldsymbol{y}' = A\boldsymbol{y}$.

Solution. Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

The $y''' = 3y'' - 2y' + y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_1 - 2y_2 + 3y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix} \boldsymbol{y}$.

Theorem 138. The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

Recall from Example 132 that the solution to $y' = ay$, $y(0) = y_0$ is $y(t) = e^{at}y_0$. Here, however, At is a matrix and so we need to make sense of the matrix exponential. The definition below defines e^A by the familiar Taylor series for e^x.

Why? Because $\boldsymbol{y}'(t) = Ae^{At}\boldsymbol{y}_0 = A\boldsymbol{y}(t)$ and $\boldsymbol{y}(0) = e^{0A}\boldsymbol{y}_0 = \boldsymbol{y}_0$.

Definition 139. Let A be $n \times n$. The **matrix exponential** is

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

How to actually compute e^A? Well, this Taylor series involves the powers A^n of A. How would you compute, say, A^{100}? The answer is diagonalization! (See next example.)

Example 140. Suppose $A = PDP^{-1}$. Then, what is A^n?

Solution. First, note that $A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$.

Likewise, $A^n = PD^nP^{-1}$. (The point being that D^n is trivial to compute because D is diagonal.)

Theorem 141. Suppose $A = PDP^{-1}$. Then, $e^A = Pe^DP^{-1}$.

Why? Recall that $A^n = PD^nP^{-1}$.

$$\begin{aligned} e^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \\ &= I + PDP^{-1} + \frac{1}{2!}PD^2P^{-1} + \frac{1}{3!}PD^3P^{-1} + \cdots \\ &= P\left(I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots\right)P^{-1} = Pe^DP^{-1} \end{aligned}$$

Comment. By the same argument, if $A = PDP^{-1}$, then $f(A) = Pf(D)P^{-1}$ for any "nice" function f. Here, "nice" means that f has a convergent Taylor series $f(x) = \sum_{n \geqslant 0} a_n x^n$.

More explicitly, if $A = P\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\,P^{-1}$, then $f(A) = P\,\mathrm{diag}(f(\lambda_1), \ldots, f(\lambda_n))\,P^{-1}$.

Example 142. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, then $A^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 3^{100} \end{bmatrix}$.

Example 143. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, then $e^A = \begin{bmatrix} e^2 & 0 \\ 0 & e^3 \end{bmatrix}$.

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