Review 109. It is important that we are able to "talk" using the basic notions of linear algebra. Here is some statements (which showed up while discussing the SVD computation) that should make perfect sense. (If not, please review or check with me.)

- A is invertible if and only if the only solution to $A\boldsymbol{x}=\boldsymbol{0}$ is $\boldsymbol{x}=\boldsymbol{0}$.
- A is singular (i.e. not invertible) if and only if $\dim \operatorname{null}(A)>0$.
- The 0-eigenspace of A is $\operatorname{null}(A)$.

  This is a special case of: the λ-eigenspace of A is $\operatorname{null}(A-\lambda I)$.
- A is singular if and only if 0 is an eigenvalue of A.
- If A has λ-eigenvalue $\boldsymbol{v}$, then A^2 has λ^2-eigenvalue $\boldsymbol{v}$.

Also, recall from our discussion of least squares solutions that:

- A^TA is invertible if and only if A has full column rank.

  Full column rank means that, if A has n columns, then $\operatorname{rank}(A)=n$ (the rank cannot be larger).

Example 110. Determine the SVD of $A=\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$.

Comment. In contrast to our previous example, $\operatorname{rank}(A)=1$. It follows that A^TA has eigenvalue 0, so that 0 is a singular value of A.

Solution. $A^TA=\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$ has 10-eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and 0-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

We conclude that $V=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\Sigma=\begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$.

$\boldsymbol{u}_1=\frac{1}{\sigma_1}A\boldsymbol{v}_1=\frac{1}{\sqrt{10}}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\frac{1}{\sqrt{20}}\begin{bmatrix} 4 \\ 2 \end{bmatrix}=\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

We cannot obtain $\boldsymbol{u}_2$ in the same way because $\sigma_2=0$. Since for every vector $\boldsymbol{u}_2$, $A\boldsymbol{v}_2=\sigma_2\boldsymbol{u}_2$, we can choose $\boldsymbol{u}_2$ as we wish, as long as the columns of U are orthonormal in the end.

$\boldsymbol{u}_2=\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ (but $\boldsymbol{u}_2=\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ works just as well)

Hence, $U=\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

In summary, $A=U\Sigma V^T$ with $U=\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, $\Sigma=\begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$, $V=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Check. Do check that, indeed, $A=U\Sigma V^T$.

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Example 111. (homework)

(a) Determine the SVD of $A = \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}$.

(b) Determine the SVD of $A = \begin{bmatrix} 1 & 5 \\ -7 & 5 \end{bmatrix}$.

Solution. (final answer only)

(a) $A = U\Sigma V^T$ with $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 3\sqrt{2} & \\ & \sqrt{2} \end{bmatrix}$, $V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) $A = U\Sigma V^T$ with $U = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 4\sqrt{5} & \\ & 2\sqrt{5} \end{bmatrix}$, $V = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$.

(But keep in mind that the SVD is not quite unique, since we have some choices.)

Example 112. (homework) Determine the SVD of $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution. $A^TA = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ has 3-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and 1-eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Since $A^TA = V\Sigma^T\Sigma V^T$, we conclude that $V = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

$\boldsymbol{u}_1 = \frac{1}{\sigma_1}A\boldsymbol{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$

$\boldsymbol{u}_2 = \frac{1}{\sigma_2}A\boldsymbol{v}_2 = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\boldsymbol{u}_3$ is chosen so that the matrix U is orthogonal. For instance, $\boldsymbol{u}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

Hence, $U = \begin{bmatrix} -2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$.

In summary, $A = U\Sigma V^T$ with $U = \begin{bmatrix} -2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$.

How did we find $\boldsymbol{u}_3$? We already have the vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, and need a vector orthogonal to both.

That is, we need to find the vector spanning $\operatorname{span}\left\{\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}^\perp = \operatorname{col}\left(\begin{bmatrix} -2 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}\right)^\perp = \operatorname{null}\left(\begin{bmatrix} -2 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\right)$.

[Without the intermediate steps, can you see why the null space consists of precisely the vectors orthogonal to both $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$?]

More generally, proceeding like this, we can always fill in "missing" vectors $\boldsymbol{u}_i$ to obtain an orthonormal basis $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ that we can use as the columns of U.

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