Recall that a point (x, y) can be represented using **polar coordinates** (r, θ), where r is the distance to the origin and θ is the angle with the x-axis.

Then, $x = r\cos\theta$ and $y = r\sin\theta$.

Every complex number z can be written in **polar form** as $z = re^{i\theta}$, with $r = |z|$.

Why? By comparing with the usual polar coordinates ($x = r\cos\theta$ and $y = r\sin\theta$), it only makes sense to write $z = x + iy$ as $z = re^{i\theta}$ if $re^{i\theta} = r\cos\theta + ir\sin\theta$. This is Euler's identity:

Theorem 104. (Euler's identity) $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Why? See below for one approach to making sense of this connection between the exponential and the trigonometric functions.

Comment. The special case $\theta = \pi$ results in the enigmatic formulas $e^{\pi i} = -1$ or $e^{\pi i} + 1 = 0$, the latter relating all five of the most fundamental mathematical constants (2 is not fundamental because $2 = 1 + 1$).

Example 105. (multiplication of complex numbers) This gives a geometric interpretation of what multiplication of complex numbers means:

$$z_1 \cdot z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2)\, e^{i(\theta_1 + \theta_2)}.$$

In words, the magnitudes multiply (as for positive real numbers), and the angles add up.

In particular, what is the geometric interpretation of multiplying with i?

Solution. Multiplication with $i = 1 \cdot e^{i\pi/2}$ does not change the magnitude but adds $\pi/2$ to the angle.

In other words, multiplication with i is a 90° rotation.

Example 106. (trig identities) Euler's identity is the mother of all trig identities! Here is just two examples:

- Take the absolute value on both sides to get $\underbrace{|e^{i\theta}|^2}_{1} = |\cos\theta + i\sin\theta| = \cos^2\theta + \sin^2\theta$.
- Use $(e^{i\theta})^2 = e^{2i\theta}$ and compare

$$\begin{aligned} (e^{i\theta})^2 &= (\cos(\theta) + i\sin(\theta))^2 = (\cos^2\theta - \sin^2\theta) + 2i\cos\theta\sin\theta, \\ e^{2i\theta} &= \cos(2\theta) + i\sin(2\theta) \end{aligned}$$

to conclude $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$ and $\sin(2\theta) = 2\cos\theta\sin\theta$.

Why? One way to see why Euler's identity holds is if you recall Taylor series from Calculus II.

Every nice function $f(x)$ can be written as $\sum_{n=0}^{\infty} a_n x^n$ (this is the Taylor series around 0, and a_n are the Taylor coefficients; you might even recall that these can be obtained as $a_n = f^{(n)}(0)/n!$).

$y(x) = e^x$ is characterized by $y' = y$, $y(0) = 1$. (We will discuss differential equations more soon!)

If $y(x) = \sum_{n=0}^{\infty} a_n x^n$, then $a_0 = y(0) = 1$. Further, $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$, and we obtain from $y' = y$ that $a_n = (n+1) a_{n+1}$. We conclude $a_n = 1/n!$ (do you see how?!).

[By the way, this is called the **Frobenius method** for finding analytic solutions of linear differential equations.]

Hence, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and we can use this to also compute with complex numbers!

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^{2n}\theta^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i^{2n+1}\theta^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n+1}}{(2n+1)!} = \cos(\theta) + i\sin(\theta)$$

In the last step, we recognized the Taylor series of cos and sin. [$i^{2n} = (-1)^n$, $i^{2n+1} = (-1)^n i$]

(Which, again, you can also derive from scratch similar to how we derived the one for e^x.)

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