Review 97. fundamental theorem of algebra

Example 98. (warmup) What is $\frac{1}{2+3i}$?

Solution. $\frac{1}{2+3i} = \frac{2-3i}{(2+3i)(2-3i)} = \frac{2-3i}{13}$.

- In general, $\frac{1}{z} = \frac{\bar{z}}{z\,\bar{z}} = \frac{\bar{z}}{|z|^2}$.
- The **absolute value** of the complex number $z = x + iy$ is $|z| = x^2 + y^2 = \bar{z}z$.
- The **norm** of the complex vector $\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is $\|\boldsymbol{z}\| = \sqrt{|z_1|^2 + |z_2|^2}$.
  Note that $\|\boldsymbol{z}\|^2 = \bar{z}_1 z_1 + \bar{z}_2 z_2 = \bar{\boldsymbol{z}}^T \boldsymbol{z}$.

Definition 99.

- For any matrix A, its **conjugate transpose** is $A^* = (\bar{A})^T$.
- The **dot product** (inner product) of complex vectors is $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \boldsymbol{v}^* \boldsymbol{w}$.
- A complex $n \times n$ matrix A is **unitary** if $A^* A = I$.

Comment. A^* is also written A^H (or $A^\dagger$ in quantum mechanics) and called the Hermitian conjugate.

Comment. For real matrices and vectors, the conjugate transpose is just the ordinary transpose. In particular, the dot product is the same.

Comment. Unitary matrices are the complex version of orthogonal matrices. (A real matrix if unitary if and only if it is orthogonal.) Again, a matrix A is unitary if and only if $\langle A\boldsymbol{v}, A\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

Example 100. What is the norm of the vector $\begin{bmatrix} 1-i \\ 2+3i \end{bmatrix}$?

Solution. $\left\| \begin{bmatrix} 1-i \\ 2+3i \end{bmatrix} \right\|^2 = \begin{bmatrix} 1+i & 2-3i \end{bmatrix} \begin{bmatrix} 1-i \\ 2+3i \end{bmatrix} = |1-i|^2 + |2+3i|^2 = 2 + 13$. Hence, $\left\| \begin{bmatrix} 1-i \\ 2+3i \end{bmatrix} \right\| = \sqrt{15}$.

Example 101. Determine A^* if $A = \begin{bmatrix} 2 & 1-i \\ 3+2i & i \end{bmatrix}$.

Solution. $A^* = \begin{bmatrix} 2 & 3-2i \\ 1+i & -i \end{bmatrix}$

Let A be a real matrix. If $\boldsymbol{v}$ is a λ-eigenvector, then $\bar{\boldsymbol{v}}$ is a $\bar{\lambda}$-eigenvector.

See, for instance, Example 96. This is just a consequence of the fact that we cannot algebraically distinguish between $+i$ and $-i$.

Example 102. Show that a symmetric real matrix A must have real eigenvalues.

This statement is part of the spectral theorem.

Solution. Suppose λ is a nonreal eigenvalue with nonzero eigenvector $\boldsymbol{v}$. Then, $\bar{\boldsymbol{v}}$ is a $\bar{\lambda}$-eigenvector and, since $\lambda \neq \bar{\lambda}$, we have two eigenvectors with different eigenvalues. Our computation in Example 89, shows that these two eigenvectors must be orthogonal in the sense that $\bar{\boldsymbol{v}}^T \boldsymbol{v} = 0$. But $\bar{\boldsymbol{v}}^T \boldsymbol{v} = \boldsymbol{v}^* \boldsymbol{v} = \|\boldsymbol{v}\|^2 \neq 0$. This shows that it is impossible to have an nonzero eigenvector for a nonreal eigenvalue.

Example 103. (homework) Find a unitary matrix Q whose first column is a multiple of $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

Solution. (sketch) We need to find a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ such that $\begin{bmatrix} 1 \\ i \end{bmatrix}^* \begin{bmatrix} a \\ b \end{bmatrix} = a - ib = 0$. Choose, say, $a = i$, $b = 1$.

This leads to the unitary matrix $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$. Indeed, $Q^* Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.