Example 92. By the spectral theorem, every symmetric matrix A can be written as $A = VDV^T$ for a diagonal matrix D and an orthogonal matrix V. What about A^{-1}?

Solution. Recall that $(AB)^{-1} = B^{-1}A^{-1}$, for any two invertible matrices A, B.

If $A = VDV^T$, then $A^{-1} = (V^T)^{-1}D^{-1}V^{-1}$. Since $V^{-1} = V^T$, this simplifies to $A^{-1} = VD^{-1}V^T$.

Comment. Likewise, $A^n = VD^nV^T$.

Example 93. If $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, what is D^{-1}?

Solution. $D^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example 94. Consider the 3×3 matrix A of a reflection through a plane (containing the origin). What can we say about A?

Solution. Here's a few things:

- A has eigenvalues 1 and -1. The 1-eigenspace is precisely the plane we are reflecting through. The -1-eigenspace is 1-dimensional and orthogonal to the plane (the normal direction of the plane).
- Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ be an orthonormal basis for the plane we are reflecting through, and let $\boldsymbol{v}_3$ (the normal direction) be a unit vector orthogonal to that plane. Then the matrix V with columns $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ is orthogonal, and $A = VDV^{-1} = VDV^T$ with D the diagonal matrix with entries $1, 1, -1$.
- More simply, $A = I - 2\frac{\boldsymbol{v}_3\boldsymbol{v}_3^T}{\boldsymbol{v}_3^T\boldsymbol{v}_3}$. (Why?! See Example 86.)
- $\det(A) = -1$ (recall once more that the determinant equals the product of the eigenvalues).
- $A^2 = I$, obviously (reflecting twice isn't doing anything). In particular, $A^{-1} = A$.
- A is symmetric, because if $A = VDV^T$ then $A^T = (V^T)^T D^T V^T = VD^TV^T = VDV^T$.
- A is orthogonal, because $A^{-1} = A = A^T$.

Comment. Why is there the condition that the plane we reflect through contains the origin? A linear map $\boldsymbol{x} \mapsto A\boldsymbol{x}$ given by a matrix A must have the property that $\mathbf{0} \mapsto \mathbf{0}$ (i.e. the origin is fixed). To talk about general reflections, we would need to consider **affine maps** $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$.

Comment. Similarly, a $n \times n$ matrix corresponds to a reflection (through a hyperplane) if and only if it has a $(n-1)$-dimensional 1-eigenspace and a 1-dimensional -1-eigenspace and these two spaces are orthogonal.

Example 95. (homework) Similar to the last example, let 3×3 be the matrix A of a projection onto a plane (containing the origin). What can we say about A?

Of course, we already know a lot about projections. The point is to think about these properties from the perspective of eigenvalues and eigenvectors.

Solution. Think about the following:

- The eigenvalues of A are $0, 1, 1$. Moreover, the 2-dimensional 1-eigenspace is orthogonal to the 1-dimensional 0-eigenspace.
- $\det(A) = 0$
- Obviously, $A^2 = A$. Can you also see that from $A = PDP^{-1}$?
- Is A symmetric? Is it orthogonal? (yes, no)

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Example 96. $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ sends $\begin{bmatrix} x \\ y \end{bmatrix}$ to $J\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$. What is the geometric description of this linear map? What are the eigenvalues and eigenvectors?

Solution. Geometrically, this is a rotation by $90°$. This makes it clear that, for no vector $\boldsymbol{x}$ in $\mathbb{R}^2$, we will have $J\boldsymbol{x} = \lambda\boldsymbol{x}$; in fact, $J\boldsymbol{x}$ and $\boldsymbol{x}$ will always be orthogonal!

In other words, J does not have any real (!!) eigenvectors.

Let's go through the math to find complex eigenstuff:

The characteristic polynomial is $\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$, and so A has eigenvalues $\pm i$.

The i-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\right)$ has basis $\begin{bmatrix} i \\ 1 \end{bmatrix}$. (Indeed, $J\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i\begin{bmatrix} i \\ 1 \end{bmatrix}$.)

The $-i$-eigenspace is $\operatorname{null}\left(\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\right)$ has basis $\begin{bmatrix} -i \\ 1 \end{bmatrix}$. (The exact same, with all i replaced with $-i$.)

Let's recall some very basic facts about **complex numbers**:

- Every complex number can be written as $z = x + iy$ with real x, y.
- Here, the imaginary unit i is characterized by solving $x^2 = -1$.

  Important observation. The same equation is solved by $-i$. This means that, algebraically, we cannot distinguish between $+i$ and $-i$.

- The **conjugate** of $z = x + iy$ is $\bar{z} = x - iy$.

  Important comment. Since we cannot algebraically distinguish between $\pm i$, we also cannot distinguish between z and $\bar{z}$. This explains that, if we start with a real problem, complex quantities always show up together with their conjugate.

  For instance, in the example above we saw that $\begin{bmatrix} i \\ 1 \end{bmatrix}$ is a i-eigenvector of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

  Replacing all i's by $-i$'s, we get that $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ is a $-i$-eigenvector of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. As we knew already.

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