Example 85. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ sends $\begin{bmatrix} x \\ y \end{bmatrix}$ to $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$. What is the geometric description of this linear map? What are the eigenvalues and eigenvectors?

Solution. Geometrically, this is a reflection through the line $y = x$. Make a sketch!

This description makes it obvious that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a 1-eigenvector, and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a -1-eigenvector.

(Of course, we can also just compute these. But do make sure that this is obvious geometrically.)

Comment. Note that the language of eigenthings makes it easy to identify and construct reflections (and other geometric transformations). See next example.

Comment. Note that the determinant of A is -1. Areas are preserved but the orientation is changed.

Example 86. **(homework)** Find the 3×3 matrix A for reflecting about the plane spanned by the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ in two different ways:

(a) By writing down the diagonalization of A.

(b) By realizing that, if $\boldsymbol{n}$ is the vector orthogonal orthogonal to the plane, then reflecting $\boldsymbol{v}$ means sending it to $\boldsymbol{v} - 2(\text{projection of } \boldsymbol{v} \text{ onto } \boldsymbol{n})$.

Solution. (some details omitted)

(a) Call this matrix A. We know that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are 1-eigenvectors.

The orthogonal complement of the plane is spanned (work!) by $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. This must be a -1-eigenvector.

Thus, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$. Hence, $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

Important comment. Note that P already has orthogonal columns. We can save ourselves time by actually choosing P such that it is an orthogonal matrix. In that case, $P^{-1} = P^T$.

Indeed, $A = PDP^T$ with $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$ and D as before.

(b) Make a sketch to see why, geometrically, $\boldsymbol{v} - 2(\text{projection of } \boldsymbol{v} \text{ onto } \boldsymbol{n})$ is indeed the reflection of $\boldsymbol{v}$.

We already observed that $\boldsymbol{n} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Hence, the reflection of $\boldsymbol{v}$ is $\boldsymbol{v} - 2(\text{projection of } \boldsymbol{v} \text{ onto } \boldsymbol{n}) = \boldsymbol{v} - 2\boldsymbol{n}\frac{\boldsymbol{n} \cdot \boldsymbol{v}}{\boldsymbol{n} \cdot \boldsymbol{n}} = \boldsymbol{v} - 2\frac{\boldsymbol{n}\boldsymbol{n}^T\boldsymbol{v}}{\boldsymbol{n}^T\boldsymbol{n}} = \left(I - 2\frac{\boldsymbol{n}\boldsymbol{n}^T}{\boldsymbol{n}^T\boldsymbol{n}}\right)\boldsymbol{v}$.

Accordingly, the reflection matrix is $A = I - 2\frac{\boldsymbol{n}\boldsymbol{n}^T}{\boldsymbol{n}^T\boldsymbol{n}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \frac{2}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

Comment. In other words, we got A from subtracting 2 times the projection matrix onto $\boldsymbol{n}$ (the normal direction) from the identity matrix.

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Example 87. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ sends $\begin{bmatrix} x \\ y \end{bmatrix}$ to $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$. Without computations, what are the eigenvalues and eigenvectors?

Solution. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a 2-eigenvector, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a 3-eigenvector.

Comment. Algebraically, this looks like a very simple map. However, notice that it is not so easy to say what happens to, say, $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ geometrically. That is because two things are happening: part of that vector is scaled by 2, the other part is scaled by 3.

Example 88. What are the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 3 \\ 3 & -7 \end{bmatrix}$?

Solution. The characteristic polynomial is $\begin{vmatrix} 1-\lambda & 3 \\ 3 & -7-\lambda \end{vmatrix} = (1-\lambda)(-7-\lambda) - 9 = (\lambda+8)(\lambda-2)$, and so A has eigenvalues $-8, 2$.

The -8-eigenspace is $\operatorname{null}\left(\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

The 2-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

In summary, $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is a -8-eigenvector, and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is a 2-eigenvector.

Important observation. The eigenvectors are again orthogonal!

Example 89. Suppose A is symmetric. Show that, if $\boldsymbol{v}$ and $\boldsymbol{w}$ are eigenvectors of A with different eigenvalues, then $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \boldsymbol{0}$.

Solution. Suppose that $A\boldsymbol{v} = \lambda\boldsymbol{v}$ and $A\boldsymbol{w} = \mu\boldsymbol{w}$ with $\lambda \neq \mu$.

Then, $\lambda\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \lambda\boldsymbol{v}, \boldsymbol{w} \rangle = \langle A\boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{v}, A^T\boldsymbol{w} \rangle = \langle \boldsymbol{v}, A\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \mu\boldsymbol{w} \rangle = \mu\langle \boldsymbol{v}, \boldsymbol{w} \rangle$.

However, since $\lambda \neq \mu$, $\lambda\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \mu\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ is only possible if $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$.

In the previous example we proved that the eigenspaces of a symmetric matrix are orthogonal. One can further show that a symmetric matrix always has enough eigenvectors and all the eigenvalues are real. This is known as the spectral theorem, which we restate here

Theorem 90. (spectral theorem) A symmetric matrix A can always be diagonalized. Moreover, all eigenvalues are real and the eigenspaces are orthogonal.

Advanced comment. A matrix such that $A^TA = AA^T$ is called **normal**. In a similar spirit as in Example 89 one can show that, for normal matrices, the eigenspaces are orthogonal to each other. However, normal matrices can have complex eigenvalues.

Example 91. (homework) Diagonalize the symmetric matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ as $A = PDP^T$.

Solution. (final solution only)

$A = PDP^T$ with $P = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} \end{bmatrix}$ and $D = \begin{bmatrix} 1+2\sqrt{2} & & \\ & 1-2\sqrt{2} & \\ & & 1 \end{bmatrix}$

Comment. Note that we were asked for a diagonalization of the form $A = PDP^T$. For that, the matrix P must be orthogonal. In particular, we must normalize its columns! (Otherwise, we only have $A = PDP^{-1}$.)

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