6 The spectral theorem

Let us add $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ to our notations for the dot product: $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \boldsymbol{v}^T \boldsymbol{w} = \boldsymbol{v} \cdot \boldsymbol{w}$.

- In our story of orthogonality, the important player has been the dot product. However, one could argue that the fundamental quantity is actually the norm:
  $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \frac{1}{4}(\|\boldsymbol{v}+\boldsymbol{w}\|^2 - \|\boldsymbol{v}-\boldsymbol{w}\|^2)$. See Example 19.
- Accepting the dot product as immensely important, we see that symmetric matrices (i.e. matrices A such that $A = A^T$) are of interest.
  For any matrix A, $\langle A\boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{v}, A^T\boldsymbol{w} \rangle$.
  It follows that, a matrix A is symmetric if and only if $\langle A\boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{v}, A\boldsymbol{w} \rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.
- Similarly, let Q be an orthogonal matrix (i.e. Q is a square matrix with $Q^T Q = I$).
  Then, $\langle Q\boldsymbol{v}, Q\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \boldsymbol{w} \rangle$.
  In fact, a matrix A is orthogonal if and only if $\langle A\boldsymbol{v}, A\boldsymbol{w} \rangle = \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

Example 82. What are the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$?

Solution. The characteristic polynomial is $\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (\lambda - 4)(\lambda + 2)$, and so A has eigenvalues $4, -2$.

The 4-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The -2-eigenspace is $\operatorname{null}\left(\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

In summary, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a 4-eigenvector, and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a -2-eigenvector.

Review. The product of all eigenvalues $-2 \cdot 4 = -8$ always equals the determinant $\det(A) = 1 - 9 = -8$.

Super important observation. The eigenvectors in the previous example are orthogonal! This is actually true for every symmetric matrix.

Theorem 83. (spectral theorem) A symmetric matrix A can always be diagonalized: $A = PDP^{-1}$. Moreover, all eigenvalues are real and the matrix P can chosen to be orthogonal.

In that case, $A = PDP^T$.

More on the spectral theorem next time.

Example 84. Sketch the effect of $\boldsymbol{x} \mapsto A\boldsymbol{x}$ with $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ in the following two ways:

(a) Where are the standard basis vectors $\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ being sent? Also sketch where the square spanned by these two vectors is sent.

(b) Repeat using the orthonormal basis $\boldsymbol{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Solution. See blackboard. Of course, $A\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $A\boldsymbol{e}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. This means that the square spanned by $\boldsymbol{e}_1, \boldsymbol{e}_2$ (a square) is sent to the parallelogram spanned by $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Moreover, if we keep track of the sides, we see that the parallelogram is flipped.

In the second case, the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ just get stretched (by a factor of 4 and -2, respectively). In particular, the square spanned by $\boldsymbol{v}_1, \boldsymbol{v}_2$ is sent to a rectangle.

Important comment. The second sketch makes the geometric interpretation of the determinant ($\det(A) = -8$) plainly visible. Namely, areas get increased by a factor of 8 (the 1×1 square is mapped to a 4×2 rectangle). The negative sign indicates that the square also gets flipped.

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