Review.

- Since there was some struggle on the quiz, recall that 2x2 systems are very pleasant to solve by hand using inverse matrices.

  For instance, the solution to $\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 9 \\ 5 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$.

- If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthogonal basis of V, and $\boldsymbol{w}$ is in V, then

$$\boldsymbol{w} = \underbrace{c_1\boldsymbol{v}_1}_{\substack{\text{proj of } \boldsymbol{w} \\ \text{onto } \boldsymbol{v}_1}} + \ldots + \underbrace{c_n\boldsymbol{v}_n}_{\substack{\text{proj of } \boldsymbol{w} \\ \text{onto } \boldsymbol{v}_n}} \qquad \text{with} \quad c_j = \frac{\boldsymbol{w}\cdot\boldsymbol{v}_j}{\boldsymbol{v}_j\cdot\boldsymbol{v}_j}.$$

  In other words, $\boldsymbol{w}$ decomposes as the sum of its projections onto each basis vector.

Example 60. Determine the orthogonal projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Solution. The orthogonal projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is $\dfrac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Example 61. Express $\underbrace{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}}_{\boldsymbol{x}}$ in terms of the basis $\underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\boldsymbol{v}_1}, \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{v}_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{v}_3}$.

Solution. Because $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ is an orthogonal basis of $\mathbb{R}^3$, we get:

$$\begin{aligned}
\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} &= c_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \underbrace{\frac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\text{projection of } \boldsymbol{x} \text{ onto } \boldsymbol{y}_1} + \underbrace{\frac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{projection of } \boldsymbol{x} \text{ onto } \boldsymbol{y}_2} + \underbrace{\frac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{projection of } \boldsymbol{x} \text{ onto } \boldsymbol{y}_3} \\
&= \frac{-4}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{10}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{4}{1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}$$

Because we spelled out all the details this looks more involved than it is. We only computed 6 dot products!

Note. Of course, this again features $-2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, the orthogonal projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Alternative. Of course, we could have solved $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ to also find $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}$.

The numbers are particularly easy here but in general, to find this solution, we have to go through the entire process of Gaussian elimination. On the other hand, if we have an orthogonal basis, the former approach requires less work, because it is just computing a few dot products.

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Example 62. (homework) Express $\begin{bmatrix}3\\7\\4\end{bmatrix}$ in terms of the basis $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$.

Solution. Note that this is an orthonormal basis of $\mathbb{R}^3$. Hence, we only need to compute the dot products

$$\begin{bmatrix}3\\7\\4\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \frac{-4}{\sqrt{2}}, \quad \begin{bmatrix}3\\7\\4\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} = \frac{10}{\sqrt{2}}, \quad \begin{bmatrix}3\\7\\4\end{bmatrix} \cdot \begin{bmatrix}0\\0\\1\end{bmatrix} = 4,$$

to find that, just as in Example 61,

$$\begin{bmatrix}3\\7\\4\end{bmatrix} = \frac{-4}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix} + \frac{10}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} + 4\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

The point is that, because the basis was orthonormal, we only had to compute 3 dot products compared to 6 in the previous example.

Comment. Working by hand, the square roots are cumbersome. For a computer working numerically, they make no difference whatsoever.

Example 63. (homework) Express $\begin{bmatrix}3\\7\\4\end{bmatrix}$ in terms of the basis $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$.

Solution. That's trivial, of course:

$$\begin{bmatrix}3\\7\\4\end{bmatrix} = 3\begin{bmatrix}1\\0\\0\end{bmatrix} + 7\begin{bmatrix}0\\1\\0\end{bmatrix} + 4\begin{bmatrix}0\\0\\1\end{bmatrix}$$

But note that the coefficients are

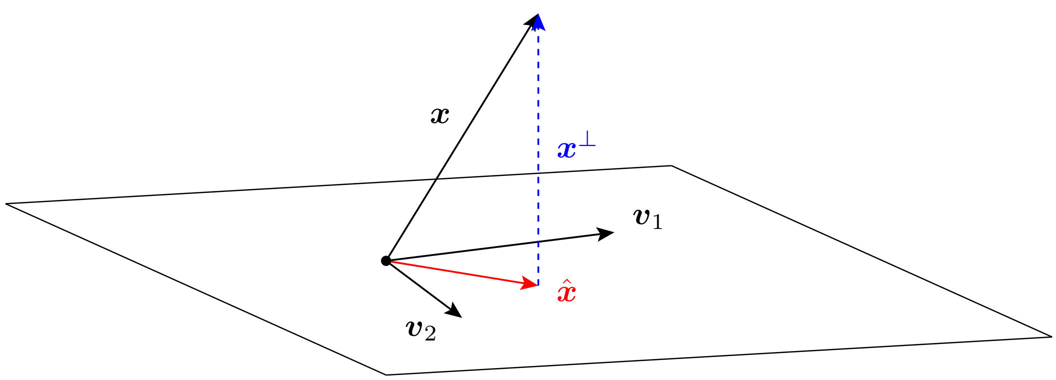
$$\begin{bmatrix}3\\7\\4\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} = 3, \quad \begin{bmatrix}3\\7\\4\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix} = 7, \quad \begin{bmatrix}3\\7\\4\end{bmatrix} \cdot \begin{bmatrix}0\\0\\1\end{bmatrix} = 4.$$

5.3 Orthogonal projection on subspaces revisited

Let W be a subspace of $\mathbb{R}^n$, and $W^\perp$ the orthogonal complement of W.

Then, each $\boldsymbol{x}$ in $\mathbb{R}^n$ can be uniquely written as

$$\boldsymbol{x} = \underbrace{\hat{\boldsymbol{x}}}_{\text{in } W} + \underbrace{\boldsymbol{x}^\perp}_{\text{in } W^\perp}.$$



- $\hat{\boldsymbol{x}}$ is the orthogonal projection of $\boldsymbol{x}$ onto W. (I.e. $\hat{\boldsymbol{x}}$ is the point in W closest to $\boldsymbol{x}$.)
- $\boldsymbol{x}^\perp$ is the orthogonal projection of $\boldsymbol{x}$ onto $W^\perp$.

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To compute the orthogonal projection $\hat{\boldsymbol{b}}$ of $\boldsymbol{b}$ onto W, we have two options:

- **(using least squares)** Write $W = \operatorname{col}(A)$, where the columns of A are a basis of W. Then, $\hat{\boldsymbol{b}} = A\hat{\boldsymbol{x}}$ where $\hat{\boldsymbol{x}}$ is the least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$ (i.e. $A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$).
- **(using orthogonality)** If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ is an orthogonal basis of W, then

$$\hat{\boldsymbol{b}} = \left(\frac{\boldsymbol{b}\cdot\boldsymbol{v}_1}{\boldsymbol{v}_1\cdot\boldsymbol{v}_1}\right)\boldsymbol{v}_1 + \ldots + \left(\frac{\boldsymbol{b}\cdot\boldsymbol{v}_m}{\boldsymbol{v}_m\cdot\boldsymbol{v}_m}\right)\boldsymbol{v}_m.$$

The next example illustrates that the second option actually is equivalent to the first option when the columns of A are an orthogonal basis of W.

Example 64. (homework) Determine the projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Solution. (using orthogonality) As in Example 61, the projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is $-2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and the projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is $4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Hence, the orthogonal projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is $-2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix}$.

Solution. (using least squares) Recall that the projection $\hat{\boldsymbol{b}}$ of $\boldsymbol{b}$ onto $\operatorname{col}(A)$ is

$$\hat{\boldsymbol{b}} = A\hat{\boldsymbol{x}}, \quad \text{with } \hat{\boldsymbol{x}} \text{ such that } A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}.$$

$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$. $\begin{bmatrix} 2 & \\ & 1 \end{bmatrix}\hat{\boldsymbol{x}} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$, $\hat{\boldsymbol{x}} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$.

Hence, the projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $\operatorname{col}(A)$ is $A\hat{\boldsymbol{x}} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix}$.

Important observation. A matrix A has orthogonal columns if and only if $A^T A$ is a diagonal matrix. (Make sure you clearly see why!! What is the meaning of the diagonal entries?)

Example 65. Continuing the previous example, what is the projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $W^\perp$?

Solution. Since the projection of $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto W is $\begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix}$, the projection onto $W^\perp$ is $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$.

Important note. Recall that we know that $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an orthogonal basis of $\mathbb{R}^3$.

Since $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ it follows that $W^\perp = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$.

Of course, we can also solve the problem by projecting $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ to get $\dfrac{\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \dfrac{10}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

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