Review.

- Vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are **linearly independent**.

  $\iff$ $c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n = \boldsymbol{0}$ only has the (trivial) solution $c_1 = c_2 = \ldots = c_n = 0$.

- Vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are a **basis** for V.

  $\iff$ $V = \mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent.

  $\iff$ Any vector $\boldsymbol{w}$ in V can be written as $\boldsymbol{w} = c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n$ in a unique way.

  The latter is the practical reason why we care so much about bases!

  V could be some abstract vector space (of polynomials or Fourier series), meaning that vectors are abstract objects and not just our usual column vectors. However, as soon as we pick a basis of V, then we can represent every (abstract) vector $\boldsymbol{w}$ by the (usual) column vector $(c_1, c_2, \ldots, c_n)^T$.

  This means all of our results can be used, too, when working with these abstract spaces!

5.2 Orthogonal bases

Theorem 56. Suppose that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are nonzero and pairwise orthogonal. Then $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent.

Proof. Suppose that

$$c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n = \boldsymbol{0}.$$

Take the dot product of $\boldsymbol{v}_1$ with both sides:

$$\begin{aligned} 0 &= \boldsymbol{v}_1 \cdot (c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n) \\ &= c_1\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 + c_2\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + \ldots + c_n\boldsymbol{v}_1 \cdot \boldsymbol{v}_n \\ &= c_1\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 = c_1\|\boldsymbol{v}_1\|^2 \end{aligned}$$

But $\|\boldsymbol{v}_1\| \neq 0$ and hence $c_1 = 0$.

Likewise, we find $c_2 = 0$, ..., $c_n = 0$. Hence, the vectors are independent. □

Comment. Note that this result is intuitively obvious: if the vectors were linearly dependent, then one of them could be written as a linear combination of the others. However, all these other vectors (and hence any combination of them) are orthogonal to it.

Definition 57. A basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ of a vector space V is an **orthogonal basis** if the vectors are (pairwise) orthogonal.

If, in addition, the basis vectors have length 1, then this is called an **orthonormal basis**.

Example 58. The standard basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an orthonormal basis for $\mathbb{R}^3$.

Armin Straub
straub@southalabama.edu

Example 59. Are the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ an orthogonal basis for $\mathbb{R}^3$? Is it orthonormal?

Solution. $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$.

So, this is an orthogonal basis.

Note that we do not need to check that the three vectors are independent. That follows from their orthogonality (see Theorem 56).

On the other hand, the vectors do not all have length 1, so that this basis is not orthonormal.

Normalize the vectors to produce an orthonormal basis.

Solution.

$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ has length $\sqrt{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} = \sqrt{2} \implies$ normalized: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ has length $\sqrt{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} = \sqrt{2} \implies$ normalized: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has length $\sqrt{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} = 1 \implies$ is already normalized: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The resulting orthonormal basis is $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be a basis of V. That means that we can write any vector $\boldsymbol{w}$ in V as a linear combination of the basis vectors, i.e. $\boldsymbol{w} = c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n$. Moreover, the values c_i are unique.

We can find the values c_i simply by solving the system $\boldsymbol{w} = c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n$.

If, in fact, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthogonal basis of V, then there is a way to determine the values c_i without solving any system! Can you see how?

Solution. Take the dot product of $\boldsymbol{v}_1$ with both sides of $\boldsymbol{w} = c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n$:

$$\begin{aligned} \boldsymbol{v}_1 \cdot \boldsymbol{w} &= \boldsymbol{v}_1 \cdot (c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n) \\ &= c_1\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 + c_2\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + \ldots + c_n\boldsymbol{v}_1 \cdot \boldsymbol{v}_n \\ &= c_1\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 \end{aligned}$$

Hence, $c_1 = \frac{\boldsymbol{v}_1 \cdot \boldsymbol{w}}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}$. In general, $c_j = \frac{\boldsymbol{v}_j \cdot \boldsymbol{w}}{\boldsymbol{v}_j \cdot \boldsymbol{v}_j}$.

Important observation. $c_1\boldsymbol{v}_1$ is the orthogonal projection of $\boldsymbol{w}$ onto $\boldsymbol{v}_1$.

In conclusion, we have found the following:

If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthogonal basis of V, and $\boldsymbol{w}$ is in V, then

$$\boldsymbol{w} = \underbrace{c_1\boldsymbol{v}_1}_{\substack{\text{proj of } \boldsymbol{w} \\ \text{onto } \boldsymbol{v}_1}} + \ldots + \underbrace{c_n\boldsymbol{v}_n}_{\substack{\text{proj of } \boldsymbol{w} \\ \text{onto } \boldsymbol{v}_n}} \qquad \text{with} \quad c_j = \frac{\boldsymbol{w} \cdot \boldsymbol{v}_j}{\boldsymbol{v}_j \cdot \boldsymbol{v}_j}.$$

In other words, $\boldsymbol{w}$ decomposes as the sum of its projections onto each basis vector.

Note. If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthonormal basis, then this simplifies to $c_j = \boldsymbol{w} \cdot \boldsymbol{v}_j$.

Armin Straub
straub@southalabama.edu