

## 3 The fundamental theorem in two acts

**Example 25. (warmup)**  $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

Note that this means that the system of equations  $\begin{matrix} x_1 + 2x_2 = 1 \\ 2x_1 + 4x_2 = -3 \\ 3x_1 + 6x_2 = 0 \end{matrix}$  can also be written as  $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$ .

[This was the motivation for introducing matrix-vector multiplication.]

In the same way, any system can be written as  $Ax = b$ , where  $A$  is a matrix and  $b$  a vector. In particular, this makes it obvious that:

$$Ax = b \text{ is consistent} \iff b \text{ is in } \text{col}(A)$$

$\text{col}(A)$  is one of the four **fundamental subspaces** associated with the matrix  $A$ .

**Theorem 26. (Fundamental Theorem of Linear Algebra, Part I)**

Let  $A$  be an  $m \times n$  matrix of **rank**  $r$ .

- $\dim \text{col}(A) = r$  (subspace of  $\mathbb{R}^m$ )
- $\dim \text{row}(A) = r$  (subspace of  $\mathbb{R}^n$ )  $\text{row}(A) = \text{col}(A^T)$
- $\dim \text{null}(A) = n - r$  (subspace of  $\mathbb{R}^n$ )
- $\dim \text{null}(A^T) = m - r$  (subspace of  $\mathbb{R}^m$ )

**Example 27.** Let  $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$ . Determine bases for all four fundamental subspaces.

**Solution.** Make sure that, for such a simple matrix, you can see all of these that at a glance!

$$\text{col}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}, \text{row}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}, \text{null}(A) = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}, \text{null}(A^T) = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$$

**Example 28. (important observation)** For  $A$  as in the previous example, what do you notice about the basis vectors for  $\text{row}(A)$  and  $\text{null}(A)$ ? What about  $\text{col}(A)$  and  $\text{null}(A^T)$ ?

**Solution.** The basis vectors for  $\text{row}(A)$  and  $\text{null}(A)$  are orthogonal!  $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$

The same is true for the basis vectors for  $\text{col}(A)$  and  $\text{null}(A^T)$ :  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = 0$  and  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = 0$

Vectors in  $\text{null}(A)$  are orthogonal to vectors in  $\text{row}(A)$ .  
For short,  $\text{null}(A)$  is orthogonal to  $\text{row}(A)$ .

**Why?** Suppose that  $x$  is in  $\text{null}(A)$ . That is,  $Ax = 0$ .  
But think about what  $Ax = 0$  means (row-product rule).  
It means that the inner product of every row with  $x$  is zero.  
But that implies that  $x$  is orthogonal to the row space.

**Definition 29.** As done in the observation above, we say that two subspaces  $V$  and  $W$  of  $\mathbb{R}^n$  are **orthogonal** if and only if every vector in  $V$  is orthogonal to every vector in  $W$ .

The **orthogonal complement** of  $W$  is the space  $W^\perp$  of all vectors that are orthogonal to  $W$ .

**Exercise.** Show that the orthogonal complement is indeed a vector space.

**Example 30.** Let  $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$ . Check that  $\text{null}(A)$  and  $\text{row}(A)$  are orthogonal complements.

**Solution.**  $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix} \xrightarrow{\substack{R_2 - 2R_1 \Rightarrow R_2 \\ R_3 - 3R_1 \Rightarrow R_3}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow{R_3 - \frac{3}{2}R_2 \Rightarrow R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Hence,  $\text{null}(A) = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}$ ,  $\text{row}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

The basis vectors are orthogonal because

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 0, \quad \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0.$$

**Note.** Because  $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$  is orthogonal to both basis vectors, it is orthogonal to every vector in the row space.

Vectors in  $\text{row}(A)$  are of the form  $\mathbf{v} = a \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ . Then,  $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \mathbf{v} = a \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$ .

**Conclusion.** Hence,  $\text{null}(A)$  and  $\text{row}(A)$  are indeed orthogonal spaces.

In fact,  $\text{null}(A)$  and  $\text{row}(A)$  are orthogonal complements.

That is because  $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  are orthogonal, hence independent, and thus a basis of all of  $\mathbb{R}^3$ .

**Theorem 31. (Fundamental Theorem of Linear Algebra, Part II)**

- $\text{null}(A)$  is orthogonal to  $\text{row}(A)$ . (both subspaces of  $\mathbb{R}^n$ )

Note that  $\dim \text{null}(A) + \dim \text{row}(A) = n$ .

Hence, the two spaces are orthogonal complements.

- $\text{null}(A^T)$  is orthogonal to  $\text{col}(A)$ .

Again, the two spaces are orthogonal complements.

**Note.** The second part is just the first part with  $A$  replaced by  $A^T$ .

**Example 32. (homework)** Determine bases for all four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 1 \\ 3 & 6 & 0 & 1 \end{bmatrix}.$$

Verify all parts of the Fundamental Theorem, especially that  $\text{null}(A)$  and  $\text{row}(A)$  (as well as  $\text{null}(A^T)$  and  $\text{col}(A)$ ) are orthogonal complements.

**Partial solution.** One can almost see that  $\text{rank}(A) = 3$ . Hence, the dimensions of the fundamental subspaces are ...