Example 13. (homework) Diagonalize, if possible, the matrices

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 0 & 2 & 0 \\ 1 & 4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution. For instance, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & -4 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & & \\ & 2 & \\ & & 2 \end{bmatrix}$. B is not diagonalizable.

For instance, $C = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}$.

Example 14. If $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, then its **transpose** is $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

2 Orthogonality

2.1 The inner product and distances

Definition 15. The **inner product** (or **dot product**) of $\boldsymbol{v}$, $\boldsymbol{w}$ in $\mathbb{R}^n$:

$$\boldsymbol{v} \cdot \boldsymbol{w} \;=\; \boldsymbol{v}^T \boldsymbol{w} = v_1 w_1 + \ldots + v_n w_n.$$

Because we can think of this as a special case of the matrix product, it satisfies the basic rules like associativity and distributivity.

In addition: $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$.

Example 16. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = 2 - 2 + 12 = 12$

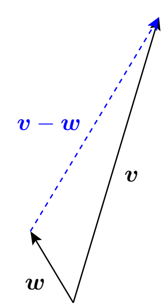
Definition 17.

- The **norm** (or **length**) of a vector $\boldsymbol{v}$ in $\mathbb{R}^n$ is

$$\|\boldsymbol{v}\| \;=\; \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + \ldots + v_n^2}.$$

- The **distance** between points $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ is

$$\mathrm{dist}(\boldsymbol{v}, \boldsymbol{w}) \;=\; \|\boldsymbol{v} - \boldsymbol{w}\|.$$



Example 18. For instance, in $\mathbb{R}^2$, $\mathrm{dist}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \left\| \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix} \right\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Example 19. Write $\|\boldsymbol{v} - \boldsymbol{w}\|^2$ as a dot product, and multiply it out.

Solution. $\|\boldsymbol{v} - \boldsymbol{w}\|^2 = (\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|^2 - 2\boldsymbol{v} \cdot \boldsymbol{w} + \|\boldsymbol{w}\|^2$

Comment. This is a vector version of $(x - y)^2 = x^2 - 2xy + y^2$.

The reason we were careful and first wrote $-\boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v}$ before simplifying it to $-2\boldsymbol{v} \cdot \boldsymbol{w}$ is that we should not take rules such as $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$ for granted. For instance, for the cross product $\boldsymbol{v} \times \boldsymbol{w}$, that you may have seen in Calculus, we have $\boldsymbol{v} \times \boldsymbol{w} \neq \boldsymbol{w} \times \boldsymbol{v}$ (instead, $\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$).

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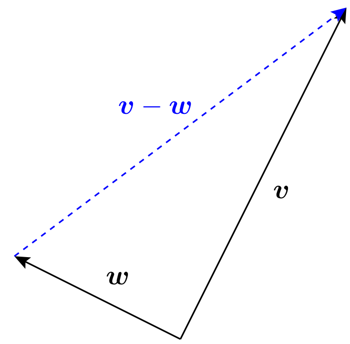
2.2 Orthogonal vectors

Definition 20. $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ are **orthogonal** if

$$\boldsymbol{v} \cdot \boldsymbol{w} = 0.$$



Why? How is this related to our understanding of right angles?

Pythagoras!

$\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal

$\iff \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = \underbrace{\|\boldsymbol{v}-\boldsymbol{w}\|^2}_{\substack{=\|\boldsymbol{v}\|^2 - 2\boldsymbol{v}\cdot\boldsymbol{w} + \|\boldsymbol{w}\|^2 \\ \text{(by previous example)}}}$

$\iff -2\boldsymbol{v}\cdot\boldsymbol{w} = 0$

$\iff \boldsymbol{v}\cdot\boldsymbol{w} = 0$

Example 21. Find vectors orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Solution. Two possibilities are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

There is many more possibilities, of course, but all of these can be obtained as linear combinations of the two vectors we already have.

For instance, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

This is not surprising: we are working in 3-dimensional space and already have 1 vector. The vectors orthogonal to it lie in a $3-1=2$-dimensional space.

Professionally speaking, the **orthogonal complement** of $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$ is $\operatorname{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

[Note how the dimensions add up to the dimension of the entire space: $1+2=3$.]

Example 22. (homework) Determine the orthogonal complement of $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$.

Solution. A possible final answer is $\operatorname{span}\left\{\begin{bmatrix} -2 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix}\right\}$. Can you find a "nicer" basis?

Example 23. The four **fundamental subspaces** associated with the matrix A are

$$\operatorname{col}(A), \quad \operatorname{row}(A), \quad \operatorname{null}(A), \quad \operatorname{null}(A^T).$$

Note that $\operatorname{row}(A) = \operatorname{col}(A^T)$.

Definition 24. $\operatorname{null}(A^T)$ is the **left null space** of A.

Why that name? Recall that, by definition $\boldsymbol{x}$ is in $\operatorname{null}(A) \iff A\boldsymbol{x} = \boldsymbol{0}$.

Likewise, $\boldsymbol{x}$ is in $\operatorname{null}(A^T) \iff A^T\boldsymbol{x} = \boldsymbol{0} \iff \boldsymbol{x}^T A = \boldsymbol{0}$.

[Recall that $(AB)^T = B^T A^T$. In particular, $(A^T\boldsymbol{x})^T = \boldsymbol{x}^T A$, which is what we used in the last equivalence.]

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